Massachusetts Institute of Technology

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6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Problem Set 5 Solutions[1]

Problem 5.1

$y(t) \equiv a$ IS AN EQUILIBRIUM SOLUTION OF THE DIFFERENTIAL EQUATION

$$y^{(3)}(t) + \ddot{y}(t) + \dot{y}(t) + 2\sin(y(t)) = 2\sin(a),$$

WHERE $a \in \mathbf{R}$ AND $y^{(3)}$ DENOTES THE THIRD DERIVATIVE OF y. FOR WHICH VALUES OF $a \in \mathbf{R}$ IS THIS EQUILIBRIUM LOCALLY EXPONENTIALLY STABLE?

The linearized equations for small $\delta(t) = y(t) - a$ are given by

$$\delta^{(3)}(t) + \ddot{\delta}(t) + \dot{\delta}(t) + 2\cos(a)\delta(t) = 0.$$

The linearized system is asymptotically stable if and only if $0 < \cos(a) < 0.5$. Hence the equilibrium $y(t) \equiv a$ of the original system is locally exponentially stable if and only if $0 < \cos(a) < 0.5$.

Problem 5.2

IN ORDER TO SOLVE A QUADRATIC MATRIX EQUATION $X^2 + AX + B = 0$, WHERE A, B ARE GIVEN n-BY-n MATRICES AND X IS AN n-BY-n MATRIX TO BE FOUND, IT IS PROPOSED TO USE AN ITERATIVE SCHEME

$$X_{k+1} = X_k^2 + AX_k + X_k + B.$$

ASSUME THAT MATRIX X_* SATISFIES $X_*^2 + AX_* + B = 0$. WHAT SHOULD BE REQUIRED OF THE EIGENVALUES OF X_* AND $A + X_*$ IN ORDER TO GUARANTEE THAT $X_k \to X_*$

[1] Version of November 12, 2003

exponentially as $k \to \infty$ when $\|X_0 - X_*\|$ is small enough? You are allowed to use the fact that matrix equation

$$ay + yb = 0,$$

where a, b, y are n-by-n matrices, has a non-zero solution y if and only if $\det(sI - a) = \det(sI + b)$ for some $s \in \mathbf{C}$.

The task at hand is to verify whether the equilibrium $X = X_*$ of the map

$$X \mapsto F(X) = X^2 + AX + X + B$$

is locally exponentially stable. Since

$$F(X + \Delta) = (X + \Delta)^2 + A(X + \Delta) + X + \Delta + B = F(X) + (X + A)\Delta + \Delta(X + I) + \Delta^2,$$

the differential dF of F at X_* is the linear transformation on the set of n-by-n matrices defined by

$$dF(\Delta) = (X_* + A)\Delta + (X_* + I)\Delta.$$

According to the standard theorem on analysis via linearization, the equilibrium is locally exponentially stable if and only if dF has no eigenvalues z with $|z| \geq 1$. Equivalently, the equation $dF(\Delta) = z\Delta$ should have no non-zero solutions Δ for all $|z| \geq 1$. According to the criterion mentioned in the problem formulation, this is true if and only if the sum of an eigenvalue of $X_* + A$ and an eigenvalue of $X_* + I - z$ is not zero for $|z| \geq 0$. Equivalently, all pairwise sums of eigenvalues of $X_* + A$ with eigenvalues of X_* should lie withing the open disc of radius one centered at -1.

Problem 5.3

Use the Center manifold theory to prove local asymptotic stability of the equilibrium at the origin of the Lorentz system

$$\begin{cases} \dot{x} = -\beta x + yz, \\ \dot{y} = -\sigma y + \sigma z, \\ \dot{z} = -yx + \rho y - z, \end{cases}$$

where β, σ are positive parameters and $\rho = 1$. Estimate the rate of convergence of $x(t), y(t), z(t)$ to zero.

Linearization of the Lorentz system equations around zero yields

$$\begin{cases} \dot{x} = -\beta x, \\ \dot{y} = -\sigma y + \sigma z, \\ \dot{z} = \rho y - z. \end{cases}$$

To diagonalize the linearized system, introduce the new state variables

$$x_1 = x, \quad x_2 = y - z, \quad x_3 = y + \sigma z.$$

This transforms the original nonlinear equations into

$$\begin{aligned}
\dot{x}_1 &= -\beta x_1 + \frac{(x_3 + \sigma x_2)(x_3 - x_2)}{(\sigma+1)^2}, \\
\dot{x}_2 &= -(\sigma+1)x_2 + \frac{(x_3 + \sigma x_2)x_1}{\sigma+1}, \\
\dot{x}_3 &= -\frac{\sigma(x_3 + \sigma x_2)x_1}{\sigma+1}.
\end{aligned}$$

According to the basic theorem, the central manifold of this system is defined by $x_1 = h_1(x_3)$ and $x_2 = h_2(x_3)$, where h_1, h_2 are 100 times continuously differentiable and satisfy

$$h_1(x_3) = h_{11}x_3^2 + o(x_3^2), \quad \dot{h}_1(x_3) = 2h_{11}x_3 + o(x_3), \quad h_2(x_3) = o(x_3^2),$$

together with

$$-\dot{h}_1(x_3)\frac{\sigma(x_3 + \sigma h_2(x_3))x_3}{\sigma+1} = -\beta h_1(x_3) + \frac{(x_3 + \sigma h_2(x_3))(x_3 - h_2(x_3))}{(\sigma+1)^2}.$$

Comparing the second order terms on both sides yields

$$h_{11} = \frac{1}{\beta(\sigma+1)^2}.$$

Substituting this into the third system equation yields

$$\dot{x}_3 = -\frac{\sigma}{\beta(\sigma+1)^3}x_3^3 + o(x_3^3),$$

which means that the center manifold system dynamics is asymptotically stable. Hence the equilibrium at the origin is locally asymptotically stable.

Problem 5.4

Check local asymptotic stability of the periodic trajectory $y(t) = \sin(t)$ of system

$$\ddot{y}(t) + \dot{y}(t) + y^3 = -\sin(t) + \cos(t) + \sin^3(t).$$

The linearized system equations for small $\delta(t) = y(t) - \sin(t)$ are given by

$$\ddot{\delta}(t) + \dot{\delta}(t) + 3\sin^2(t)\delta(t) = 0,$$

or, equivalently

$$\dot{x}(t) = A(t)x(t), \quad A(t) = \begin{bmatrix} 0 & 1 \\ -3\sin^2(t) & -1 \end{bmatrix},$$

where

$$x(t) = \begin{bmatrix} \delta(t) \\ \dot{\delta}(t) \end{bmatrix}.$$

The evolution matrix of the linear system over its period π, calculated numerically using the MATLAB code[2]

```
M=eye(2);
T=pi;
for k=1:n,
    M=expm([0 1;-3*sin(k*T/n) -1]*(T/n))*M;
end
```

is given by

$$M \approx \begin{bmatrix} -0.2995 & -0.2362 \\ 0.0986 & -0.0665 \end{bmatrix},$$

and has eigenvalues well within the unit circle. Hence, the periodic solution is locally exponentially stable.

Problem 5.5

FIND ALL VALUES OF PARAMETER $a \in \mathbf{R}$ SUCH THAT EVERY SOLUTION $x : [0, \infty) \mapsto \mathbf{R}^2$ OF THE ODE

$$\dot{x}(t) = \epsilon \begin{bmatrix} \cos(2t) & a \\ \cos^4(t) & \sin^4(t) \end{bmatrix} x(t)$$

CONVERGES TO ZERO AS $t \to \infty$ WHEN $\epsilon > 0$ IS A SUFFICIENTLY SMALL CONSTANT.

Since the integral of trace of

$$A(t) = \begin{bmatrix} \cos(2t) & a \\ \cos^4(t) & \sin^4(t) \end{bmatrix}$$

over its period π is positive, there are no $a \in \mathbf{R}, \epsilon > 0$ for which all solutions converge to zero as $t \to +\infty$.

A more interesting case takes place when $A(t)$ is replaced with

$$A_1(t) = \begin{bmatrix} \cos(2t) & a \\ \cos^4(t) & -\sin^4(t) \end{bmatrix}.$$

[2]See the attached file `hw5_4_624_2003.m`

Then the average of $A_1(t)$ over the period equals

$$\bar{A} = \begin{bmatrix} 0 & a \\ 3/8 & -3/8 \end{bmatrix}.$$

When $a < 0$, this is a Hurwitz matrix, which, according to the averaging theorem, guarantees asymptotic stability for sufficiently small $\epsilon > 0$. When $a = 0$, the original equations yield

$$\dot{x}_1(t) = \cos(2t)x_1(t),$$

and hence $x_1(t)$ does not converge to zero as $t \to \infty$ when $x_1(0) \neq 0$. When $a > 0$, $\bar{A}$ has eigenvalues with positive real part. Repeating the arguments from the proof of Theorem 10.2 shows that the evolution matrix of the system will have eigenvalues outside of the closed unit disc for all sufficiently small $\epsilon > 0$.