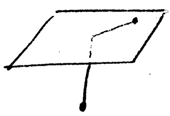
Session 6

2-032
9,27-1



Recall:

- $E = T + V = m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 + mgr = E_0 = \text{Const}$

- $H_0 = m r^2 \dot{\phi} = \text{Const} \Rightarrow \boxed{\dot{\phi} = \frac{H_0}{m r^2}}$ $(*)$

Note: φ is a cyclic coordinate (ignorable)

$$\frac{\partial E}{\partial \varphi} = 0$$

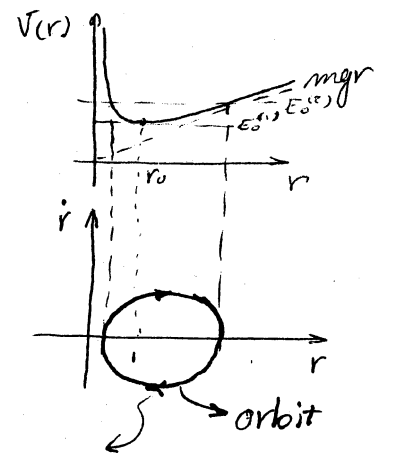
when such a coordinate present,

$\#$ DOF can be reduced by one $\Rightarrow$ reduced mechanical system

In present case, use $(*)$ to obtain reduced energy

$$E = \underbrace{m\dot{r}^2}_{T(\dot{r})} + \underbrace{\frac{H_0^2}{2m} \cdot \frac{1}{r^2} + mgr}_{V(r)} = E_0$$

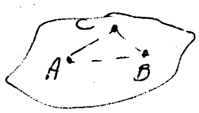
$$\dot{r} = \sqrt{\frac{1}{m}\left(E_0 - V(r)\right)}$$



the Only Consistence direction

Rigid Body Dynamics

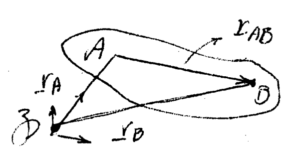
(1)



$|\underline{r}_{AB}| = \text{Const}$

(2) $\#$ DOF $= 6$

(3) Velocities at different points of a rigid body



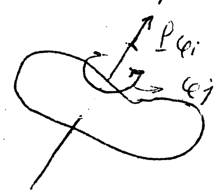
$\underline{r}_B = \underline{r}_A + \underline{r}_{AB}$

$\underline{V}_B = \underline{V}_A + \dot{\underline{r}}_{AB}$

It turns out that there exist a unique vector $\underline{\omega}$ (Angular Velocity of the rigid body) such that

$$\boxed{\underline{V}_B = \underline{V}_A + \underline{\omega} \times \underline{r}_{AB}}$$ for all A & B.

(2) If the rotation of the rigid body can be instantaneously decomposed to a finite # of rotation about "well-understood" fixed axes, then the angular velocities defined for those rotations, then $\underline{\omega}$ is just the sum of these angular velocities.



$$\Rightarrow \underline{\omega} = \sum_{i=1}^{n} \underline{\omega}_i \Rightarrow \text{the order of summation is unimportant}$$

Surprising, because finite rotation in 3d don't commute

to prove (1), note that instantaneously, B performs an instantaneous rotation about A

In general Rotation in 3D about a fixed point can be described through matrix multiplication.

$$\underline{e}(t) = \underline{\underline{R}}(t)\,\underline{e}_0$$ where $R(t)$ is a $\boxed{\text{proper orthogonal matrix}}$

Main properties of such matrices

(a) Preserve length $|\underline{e}(t)|^2 = |\underline{e}_0|^2$ or $\langle \underline{\underline{R}}(t)\underline{e}_0, \underline{\underline{R}}(t)\underline{e}_0\rangle = \langle \underline{e}_0, \underline{e}_0\rangle$

In general $\langle \underline{\underline{T}}\,\underline{a}, \underline{b}\rangle = \langle \underline{a}, \underline{\underline{T}}^{t}\,\underline{b}\rangle$

transpose of $\underline{\underline{T}}$

$$\langle \underline{e}_0, \underbrace{\underline{\underline{R}}^T(t)\,\underline{\underline{R}}(t)}_{\underline{\underline{I}}}\underline{e}_0\rangle = \langle \underline{e}_0, \underline{e}_0\rangle$$

$\underline{\underline{I}}$ (because $\underline{e}_0$ is an identity)

here $\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \underline{\underline{R}}^{-1} = \underline{\underline{R}}^T$

$$\det(R^T)\cdot\det(R) = 1 \Rightarrow |\det(R)| = 1$$

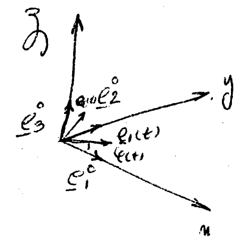
b) Preserve orientation of vectors

$$\Rightarrow \det \underline{\underline{R}} > 0 \Rightarrow \det(R(t)) = 1$$

Example

$$\underline{\underline{R}}(t) = \left(\begin{pmatrix} \cos\varphi(t) \\ \sin\varphi(t) \\ 0 \end{pmatrix} \begin{pmatrix} -\sin\varphi(t) \\ \cos\varphi(t) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\qquad\quad \underline{e}_1(t) \qquad\quad \underline{e}_2(t) \qquad \underline{e}_3(t)$$



Using the above, fixing A we obtain

$$\frac{d}{dt}\left\{ \underline{r}_{AB}(t) = \underline{\underline{R}}(t)\, \underline{r}_{AB}(0) \right\}$$

$$\dot{\underline{r}}_{AB} = \dot{\underline{\underline{R}}}\, \underline{r}_{AB}(0)$$

Note: $\underline{\underline{R}}\, \underline{\underline{R}}^t = I \quad / \frac{d}{dt}$

$$\dot{\underline{\underline{R}}}\, \underline{\underline{R}}^t + \underline{\underline{R}}\, \dot{\underline{\underline{R}}}^t = 0$$

$$\Rightarrow \dot{\underline{\underline{R}}} = -\underline{\underline{R}}\, \dot{\underline{\underline{R}}}^t \underline{\underline{R}}$$

$$\Rightarrow \dot{\underline{r}}_{AB} = \underbrace{-\underline{\underline{R}}\, \dot{\underline{\underline{R}}}^t}_{\underline{\underline{\Omega}}^A(t)} \underbrace{\underline{\underline{R}}\, \underline{r}_{AB}(0)}_{\underline{r}_{AB}(t)} \qquad \Rightarrow \left[\underline{\underline{\Omega}}^A(t)\right]^T = -\underline{\underline{\Omega}}^A(t)$$

$$= \underline{\underline{\Omega}}^A(t)\, \underline{r}_{AB}(t) \qquad \Rightarrow \underline{\underline{\Omega}}^A(t) \text{ skew symmetric}$$

By Assignment #2, for any 3D skew symmetric matrix Ω^A, there exists a 3D vector $\underline{\omega}^A$ such that $\underline{\underline{\Omega}}^A \underline{r} = \underline{\omega}^A \times \underline{r}$

$$\dot{\underline{r}}_{AB}(t) = \underline{\underline{\Omega}}^A(t)\, \underline{r}_{AB}(t)$$

$$= \underline{\omega}_A(t) \times \underline{r}_{AB}(t) \qquad \text{but } \dot{\underline{r}}_{AB} = \underline{v}_B - \underline{v}_A$$

$$\Rightarrow \boxed{\underline{v}_B = \underline{v}_A + \underline{\omega}_A \times \underline{r}_{AB}}$$