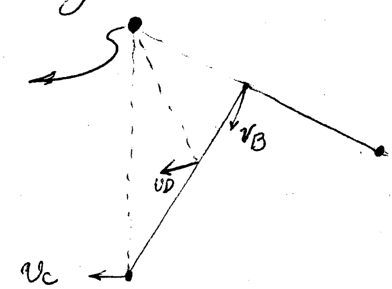
* in 2D if we know v_B and v_A that is enough to determine ω. but in 3D we need at least to know the velocity of 3 points

instantaneous Center of Rotation

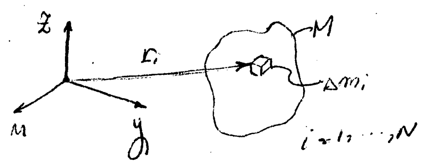


You can get the direction of velocity by having the instantaneous Center of Rotation

~~Kinetics~~ in 3 dim, there is an axis of Rotation

Kinetics of Rigid bodies

(1) Linear momentum principle



Define linear momentum as $P = \lim_{\substack{N \to \infty \\ \Delta m_i \to 0}} \sum_{i=1}^{N} \dot{r}_i \, \Delta m_i = \int_M \underline{v} \, dm$

Note: $dm = \rho dV = \rho \, dx \, dy \, dz$

By taking the limit $N \to \infty$ in our discussion for systems of particles:

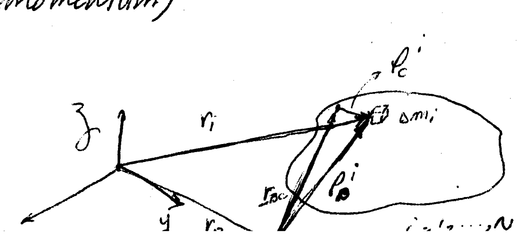
$\boxed{\underline{P} = M \underline{V}_c}$ where c is the Center of mass defined by $r_c = \frac{1}{M} \int_M \underline{r} \, dm$

By def, C is again the point for which the total mass moment vanishes

Also, $N \to \infty$ gives $\boxed{\dot{\underline{P}} = \underline{F}}$ F: resultant external force

. if $F = 0 \Rightarrow \underline{P} = \text{Const}$ (Conservation of linear momentum)

(2) Angular momentum principle



Define angular momentum $H_B = \lim_{\substack{N\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{N} \rho_B^i \times (\Delta m_i \dot{r}_i)$

$$= \int_M \rho_B \times \underline{V} \, dm$$

taking the $N \to \infty$ limit in our Calculations in Systems of particles:

$$\dot{H}_B + \underline{V}_B \times \underline{P} = \underline{M}_B$$

where $\underline{M}_B$ is the resultant external torque w.r.t. B

Special Case

if $\underline{M}_B = \underline{0}$ AND $\underline{V}_B = \underline{0}$ or $B = CM$ or $\underline{V}_B \parallel V_C$

then $\underline{H}_B = \text{Const}$ Conservation of angular momentum

How do we Compute $\underline{H}_B$?

First note that if c is the CM, then $H_B = \int_M (\underline{r}_{BC} + \rho_c) \times (\underline{V}_c + \underline{\omega} \times \underline{\rho}_c)\, dm$

$$= \underline{r}_{BC} \times (\underline{V}_c M) + \underline{r}_{BC} \times \underline{\omega} \times \int_M \rho_c \, dm + \underbrace{\int_M \rho_c \times (\underline{V}_c + \omega \times \underline{\rho}_c)\, dm}_{H_C}$$

$$\Rightarrow \boxed{\underline{H}_B = \underline{H}_C + \underline{P} \times \underline{r}_{CB}} \quad (*)$$

Similar to $V_B = V_A + \omega \times r_{AB}$

where $\boxed{\underline{H}_C = \int_M \underline{\rho}_c \times \underline{\omega} \times \underline{\rho}_c \, dm} \rightarrow$ Centroidal angular momentum

Note: for any other point A

$$(**) \quad \underline{H}_A = \underline{H}_C + \underline{P} \times \underline{r}_{CA}$$

then $(*)$ and $(**)$ $\Rightarrow$ $\boxed{\underline{H}_B = \underline{H}_A + \underline{P} \times \underline{r}_{AB}}$

Computation of $\underline{H}_C$

Note: $\rho_c \times (\underline{\omega} \times \rho_c) = (\rho_c \cdot \rho_c)\underline{\omega} - (\underline{\rho}_c \cdot \underline{\omega})\underline{\rho}_c$

$$(\underline{a} \times \underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\,\underline{b} - (\underline{a} \cdot \underline{b})\,\underline{c}$$

$$\Rightarrow \rho_c \times (\omega \times \rho_c) = \begin{pmatrix} x^2+y^2+z^2 & 0 & 0 \\ 0 & x^2+y^2+z^2 & 0 \\ 0 & 0 & x^2+y^2+z^2 \end{pmatrix} \underline{\omega} - \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} \underline{\omega}$$

(x, y, z) to CM

$$= \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \underline{\omega}$$

$\Rightarrow \underline{H}_c = \underline{\underline{I}}_c \, \underline{\omega}$

where $\underline{\underline{I}}_c$ is the <u>Centroidal moment of inertia tensor</u> defined as:

$$\underline{\underline{I}}_c = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix}$$

with $I_{xx} = \int_M (y^2+z^2)\,dm$

$I_{xy} = -\int_M xy\,dm$ $\qquad I_{yy} = \int_M (x^2+z^2)\,dm$

$I_{xz} = -\int_M xz\,dm$ $\qquad I_{zz} = \int_M (x^2+y^2)\,dm$

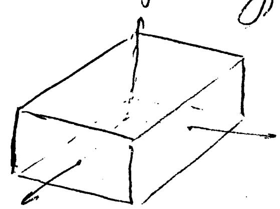
$I_{yz} = -\int_M yz\,dm$

properties of $\underline{\underline{I}}_c$

- Symmetric $\Rightarrow$ 3 real
- eigenvalues orthogonal
- eigen vectors (principal axes of inertia)

In the principal axes frame $\underline{\underline{I}}_c = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$

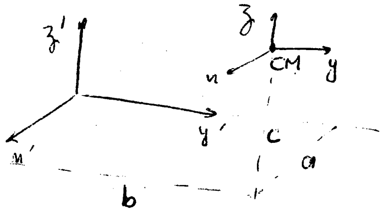
Note: axes of symmetry are Automatically principal axes



- $\underline{\underline{I}}_c$ positive definite, i.e. $I_1, I_2, I_3 > 0$ (will see later why)
- In 2D $\quad \underline{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \omega \underline{K}$

$$\Rightarrow \boxed{\underline{H}_c = I_{zz}\, \omega \underline{K}}$$

- Parallel axis theorem (xyz attached to CM)



then $\underline{\underline{I}}_O = \underline{\underline{I}}_c + M \begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}$

in 2D. $I_O = I_C + Mh^2$