Lecture 13 - Physical Explanation of Gauss Elimination

Prof. K. J. Bathe   MIT OpenCourseWare
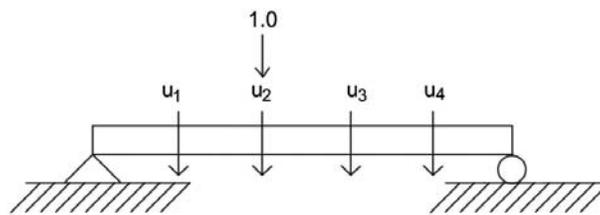
Reading assignment: Sections 8.1, 8.2

Gauss elimination is the direct method of solution, which includes:

- LDL^T factorization; sparse techniques

- Wavefront method

- Substructuring, super-element techniques, etc.

- Static condensation

Example

Reading assignment: Section 8.2.1

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let's perform Gauss elimination on this system.

Step 1: Add/subtract multiples of row 1 to the other rows to obtain:

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & -\frac{16}{5} & \frac{29}{5} & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

We used $5u_1 - 4u_2 + u_3 = 0$ to eliminate u_1 from the second and third equations.

Step 2: Add/subtract multiples of row 2 to rows 3 and 4 to obtain:

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & -\frac{20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ -\frac{5}{14} \end{bmatrix}$$

1

Step 3: Add/subtract multiples of row 3 to row 4 to obtain:

$$
\begin{bmatrix}
5 & -4 & 1 & 0 \\
0 & \frac{14}{5} & -\frac{16}{5} & 1 \\
0 & 0 & \frac{15}{7} & -\frac{20}{7} \\
0 & 0 & 0 & \frac{5}{6}
\end{bmatrix}
\begin{bmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 1 \\ \frac{8}{7} \\ \frac{7}{16}
\end{bmatrix}
$$

We have performed factorization/triangularization of the stiffness matrix K, and the forward reduction of R. u_4, u_3, u_2, u_1 can now be found in this order by performing the back-substitution.

## The Physical Process of Gauss Elimination

The process of Gauss elimination can be demonstrated physically by performing the following lab experiment. Suppose that in the laboratory the beam shown below is constructed. At the locations where the degrees of freedom are measured, clamps are fastened with force measuring devices. Now, sequentially impose unit displacements to each degree of freedom and measure the required forces in all devices. These forces correspond to the columns of the stiffness matrix.
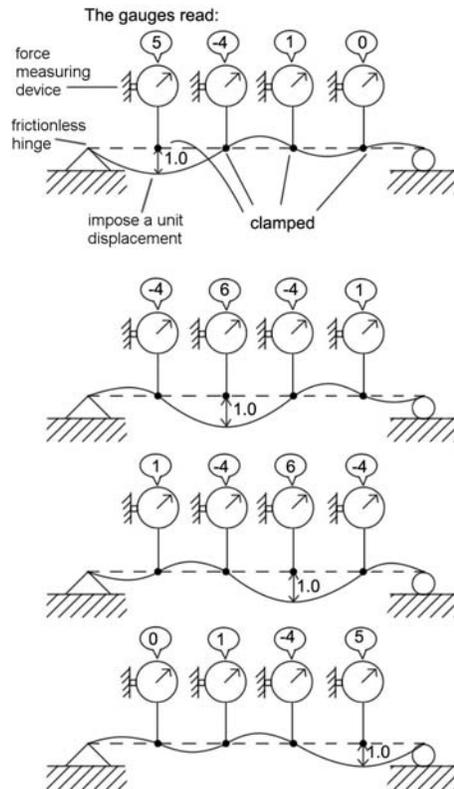
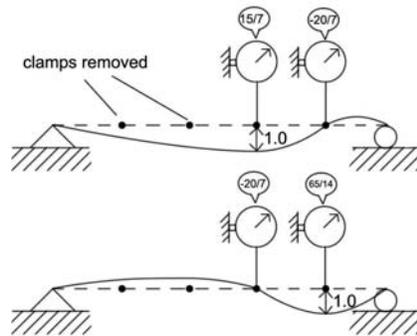By the experimental results, we obtain

$$
\begin{bmatrix}
5 & -4 & 1 & 0 \\
-4 & 6 & -4 & 1 \\
1 & -4 & 6 & -4 \\
0 & 1 & -4 & 5
\end{bmatrix}
\rightarrow K
$$

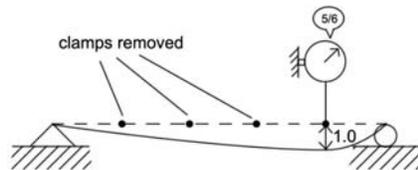Step 1: Remove clamp 1 and repeat the experiment on the same physical beam:

clamp removed
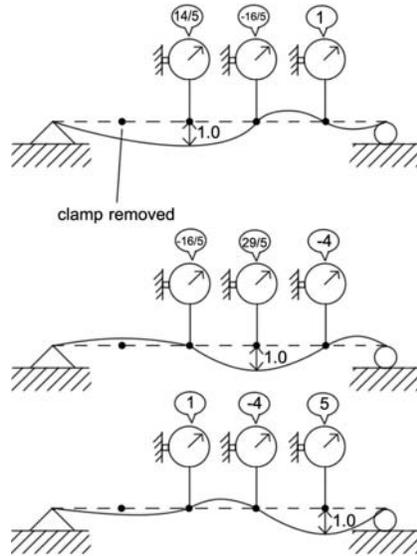
$$\begin{bmatrix} \frac{14}{5} & -\frac{16}{5} & 1 \\ -\frac{16}{5} & \frac{29}{5} & -4 \\ 1 & -4 & 5 \end{bmatrix}$$

Step 2: Remove clamp 2.

clamps removed

$$\begin{bmatrix} \frac{15}{7} & -\frac{20}{7} \\ -\frac{20}{7} & \frac{65}{14} \end{bmatrix}$$
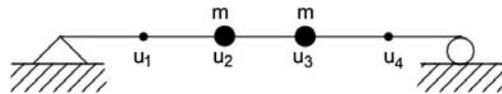
Step 3: Remove clamp 3.

clamps removed

$$\begin{bmatrix} \frac{5}{6} \end{bmatrix}$$

We note that the dials measure the stiffness after each step of Gauss elimination. Each step of Gauss elimination corresponds to the removal of a clamp.
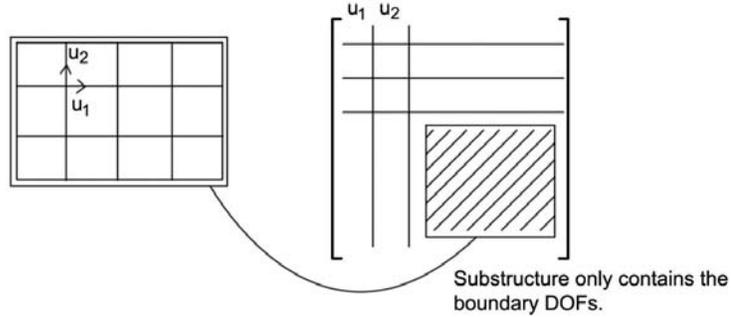
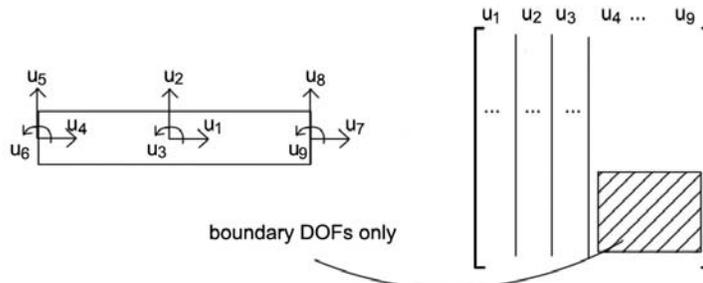# Applications

I. Static condensation

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For this system, we can work with the 2×2 matrix $\tilde{K}$ instead of the full 4×4 K. See lecture notes on dynamic analysis for details on static condensation.

II. Substructuring/Super-element techniques (2D)

Substructure only contains the boundary DOFs.

III. Beam example

boundary DOFs only

2.092 / 2.093 Finite Element Analysis of Solids and Fluids I
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