13.002

Introduction to Numerical Methods for Engineers

Solution to Problem Set 3

1. Using Gaussian elimination:

$$\begin{aligned} 2x_1 + x_2 + 4x_3 &= 16 \\ 3x_1 + 2x_2 + x_3 &= 10 \\ x_1 + 3x_2 + 3x_3 &= 16. \end{aligned}$$

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + 2x_3 &= 8 \\ \frac{1}{2}x_2 - 5x_3 &= -14 \\ \frac{5}{2}x_2 + x_3 &= 8. \end{aligned}$$

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + 2x_3 &= 8 \\ x_2 - 10x_3 &= -28 \\ 26x_3 &= 78. \end{aligned}$$

$$x_1 = 1 \quad x_2 = 2 \quad x_3 = 3$$

2.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$ we get

$$x_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

3.

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7, \quad |B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2, \quad |B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4$$

From Cramer's Rule:

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{7} \quad x_2 = \frac{|B_2|}{|A|} = \frac{2}{7} \quad x_3 = \frac{|B_3|}{|A|} = \frac{4}{7}$$