Introduction to Numerical Analysis for Engineers

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Initial Value Problems
Higher Order Differential Equations

Differential Equation

$$
\begin{aligned}
y^{(n)}(t) &= f(t, y, y', \dots y^{(n-1)}) \\
y(t_0) &= y_0 \\
y'(t_0) &= y_1 \\
&. \\
&. \\
y^{(n-1)}(t_0) &= y_{n-1}
\end{aligned}
\quad \Bigg\} \text{ Initial Conditions}
$$

Convert to 1st Order System

$$
\left.
\begin{aligned}
x_1 &= y \\
x_2 &= y' \\
x_3 &= y'' \\
&. \\
&. \\
x_n &= y^{(n-1)}
\end{aligned}
\right\} \Rightarrow
\left\{
\begin{aligned}
x_1' &= x_2 & x_1(t_0) &= y_0 \\
x_2' &= x_3 & x_2(t_0) &= y_1 \\
x_3' &= x_4 & x_3(t_0) &= y_2 \\
&. \\
&. \\
x_n' &= f(t, x1, x2, \dots x_n) & x_n(t_0) &= y_{n-1}
\end{aligned}
\right.
$$

Matrix form

$$\overline{\mathbf{x}}' = \overline{\overline{\mathbf{A}}}\overline{\mathbf{x}} + \overline{\mathbf{g}}$$

$$
\overline{\mathbf{x}} = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{Bmatrix}, \quad
\overline{\overline{\mathbf{A}}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ . & & . \\ . & & . \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}
$$

Solved using e.g. Runge-Kutta (ode45)

Boundary Value Problems
Shooting Method

Differential Equation

$$y'' = f(x, y, y')$$

$$\left.\begin{aligned} y(a) &= y_a \\ y(b) &= y_b \end{aligned}\right\} \text{Boundary Conditions}$$

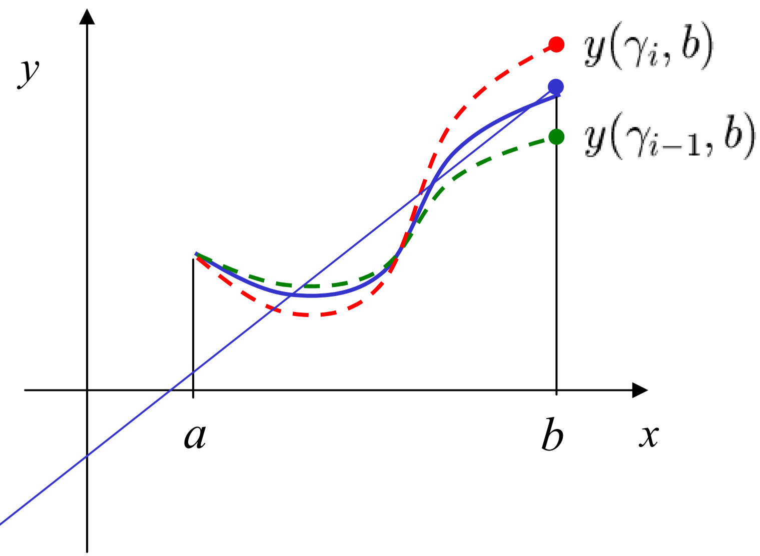
'Shooting' Method

$$\left.\begin{aligned} y'' &= f(x, y, y') \\ y(a) &= y_a \\ y'(a) &= \gamma_i \end{aligned}\right\} \rightarrow y(\gamma_i, b)$$

Initial value Problem
Solve by Runge-Kutta

'Shooting' Iteration

$$\gamma_{i+1} = \gamma_{i-1} + (\gamma_i - \gamma_{i-1}) \frac{y_b - y(\gamma_{i-1}, b)}{y(\gamma_i, b) - y(\gamma_{i-1}, b)}$$



Boundary Value Problems
Direct Finite Difference Methods

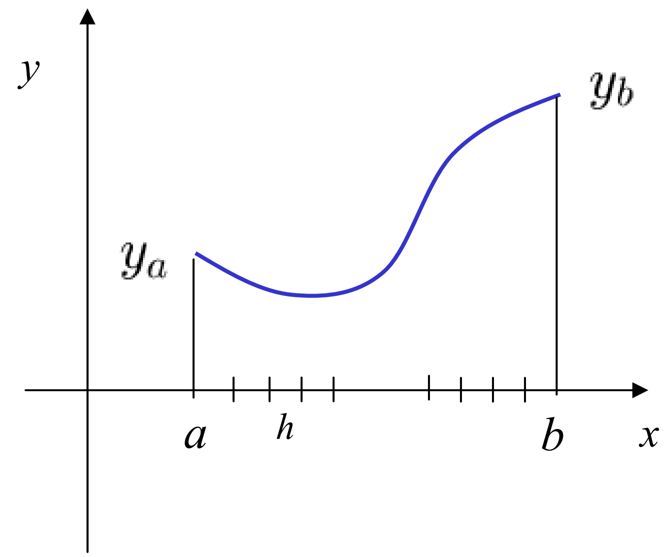
Differential Equation

$$
\begin{aligned}
y'' &= f(x, y, y') \\
y(a) &= y_a \\
y(b) &= y_b
\end{aligned}
\quad \text{Boundary Conditions}
$$

Discretization

$$h = \frac{b-a}{N}$$

$$x_n = a + nh \;, \quad n = 0, 1 \ldots N$$



Finite Differences

$$
\begin{aligned}
y_n &= y(x_n) \\
y'_n &= y'(x_n) = \frac{y_{n+1} - y_{n-1}}{2h} + O(h^2) \\
y''_n &= y''(x_n) = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + O(h^2)
\end{aligned}
$$

Boundary Value Problems
Direct Finite Difference Methods

Boundary value Problem

$$
\begin{aligned}
y'' &= f(x, y, y') \\
y(a) &= y_a \\
y(b) &= y_b
\end{aligned}
$$

Finite Differences

$$
\begin{aligned}
y_n &= y(x_n) \\
y'_n &= y'(x_n) = \frac{y_{n+1} - y_{n-1}}{2h} + O(h^2) \\
y''_n &= y''(x_n) = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + O(h^2)
\end{aligned}
$$

Substitute Finite Differences

$$
\begin{aligned}
y_{n+1} - 2y_n + y_{n-1} &= h^2 f(x_n, y_n, \frac{y_{n+1} - y_{n-1}}{2h}) \ , \quad n = 1, 2, \ldots N\text{-}1 \\
y_0 &= y_a \\
y_n &= y_b
\end{aligned}
$$

Difference Equations

$$
\begin{aligned}
-2y_1 + y_2 &= h^2 f\left(x_1, y_1, \frac{y_2 - y_a}{2h}\right) - y_a \\
y_{n-1} - 2y_n + y_{n+1} &= h^2 f\left(x_n, y_n, \frac{y_{n+1} - y_{n-1}}{2h}\right) \ , \quad n = 2, \ldots N-2 \\
y_{N-2} - 2y_{N-1} &= h^2 f\left(x_{N-1}, y_{N-1}, \frac{y_b - y_{N-2}}{2h}\right) - y_b
\end{aligned}
$$

N-1 equations, N-1 unknowns

Matrix Equations

$$
\begin{bmatrix}
-2 & 1 & & & & \\
1 & -2 & 1 & & & \\
 & & \cdot & & & \\
 & & & \cdot & & \\
 & & & 1 & -2 & 1 \\
 & & & & 1 & -2
\end{bmatrix}
\begin{bmatrix}
y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{N-1}
\end{bmatrix}
- h^2
\begin{bmatrix}
f(x_1, y_1, \frac{y_2 - y_a}{2h}) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f(x_{N-1}, y_{N-1}, \frac{y_b - y_{N-2}}{2h})
\end{bmatrix}
=
\begin{bmatrix}
-y_a \\ 0 \\ \cdot \\ \cdot \\ 0 \\ -y_b
\end{bmatrix}
$$

$$
\overline{\overline{\mathbf{A}}}\overline{\mathbf{y}} - h^2 \overline{\mathbf{f}}(\overline{\mathbf{y}}) = \overline{\mathbf{r}}
$$

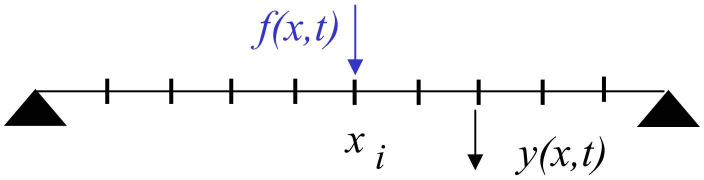
Linear Differential Equations

$$
\begin{aligned}
\overline{\overline{\mathbf{A}}}\overline{\mathbf{y}} - h^2 \overline{\overline{\mathbf{G}}}(x)\overline{\mathbf{y}} &= \overline{\mathbf{r}} \\
\left[\overline{\overline{\mathbf{A}}} - h^2 \overline{\overline{\mathbf{G}}}(x)\right]\overline{\mathbf{y}} &= \overline{\mathbf{r}}
\end{aligned}
$$

Solve using standard linear system solver

Boundary Value Problems
Finite Difference Methods

Forced Vibration of a String



Harmonic excitation

$$f(x,t) = f(x)\cos(\omega t)$$

Differential Equation

$$\frac{d^2y}{dx^2} + k^2 y = f(x)$$

Boundary Conditions

$$y(0) = 0\ ,\quad y(L) = 0$$

Finite Difference

$$\left.\frac{d^2y}{dx^2}\right|_{x_i} \simeq \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

Discrete Difference Equations

$$y_{i-1} + \left((kh)^2 - 2\right) y_i - y_{i+1} = f(x_i)h^2$$

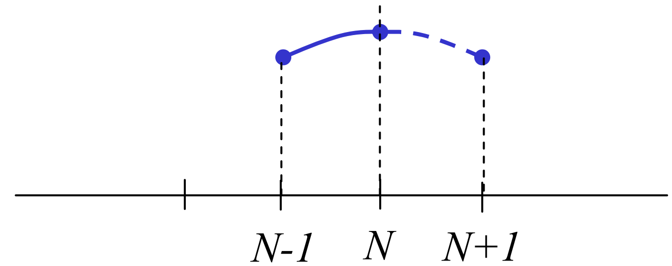
Matrix Form

$$\begin{bmatrix} (kh)^2-2 & 1 & \cdot & \cdot & \cdot\ \cdot & 0 \\ 1 & (kh)^2-2 & 1 & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & 1 & (kh)^2-2 & 1 & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot\ 1 & (kh)^2-2 \end{bmatrix} \overline{\mathbf{x}} = \begin{Bmatrix} f(x_1)h^2 \\ \cdot \\ \cdot \\ f(x_i)h^2 \\ \cdot \\ \cdot \\ f(x_n)h^2 \end{Bmatrix}$$

Tridiagonal Matrix

$kh < 1$ Symmetric, positive definite: No pivoting needed

Boundary Value Problems
Finite Difference Methods



Boundary Conditions with Derivatives

$$
\begin{aligned}
y'' - yx &= g(x) \\
y(a) &= 0 \\
y'(b) &= 0
\end{aligned}
$$

Central Difference

Difference Equations

$$
\begin{aligned}
y_0 &= 0 \\
y_{n-1} - 2y_n + y_{n+1} - h^2 y_n x_n &= h^2 g(x_n)\ , \quad n = 1, \\
y_N &= \ ?
\end{aligned}
$$

Backward Difference

$$y'(b) = 0 = \frac{y_N - y_{N-1}}{h} + O(h)$$

$$
\begin{aligned}
y_0 &= 0 \\
y_{n-1} - 2y_n + y_{n+1} - h^2 y_n x_n &= h^2 g(x_n)\ , \quad n = 1, 2, \ldots N-1 \\
y_N - y_{N-1} &= 0
\end{aligned}
$$

O(h^4)

O(h^2)

Central Difference

$$y'(b) = 0 = \frac{y_{N+1} - y_{N-1}}{2h} + O(h^2)$$

$$
\begin{aligned}
y_0 &= 0 \\
y_{n-1} - 2y_n + y_{n+1} - h^2 y_n x_n &= h^2 g(x_n)\ , \quad n = 1, 2, \ldots N-1 \\
2(y_{N-1} - y_N) - h^2 y_N x_N &= 0
\end{aligned}
$$

O(h^3)

General Boundary Conditions

$$p_0 y(b) + p_1 y'(b) = p_2$$

Finite Difference Representation

$$p_0 y_N + \frac{p_1 (y_{N+1} - y_{N-1})}{2h} = p_2$$

Add extra point - N equations, N unknowns