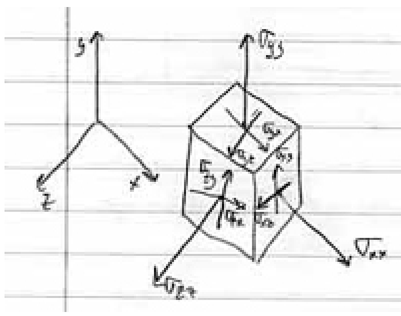
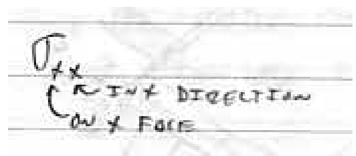
2.001 - MECHANICS AND MATERIALS I
Lecture #13
10/25/2006
Prof. Carol Livermore

Recall from last time:

$$[\sigma\ ] = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix}.$$

Diagonal terms are normal stresses.
Off diagonal terms are shear stresses.

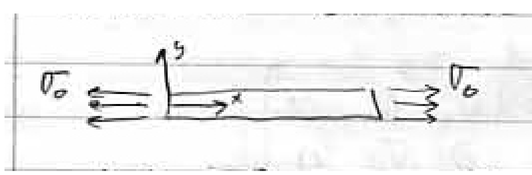




Not as bad as it seems:

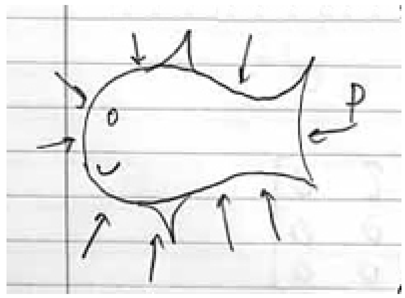
1. Linearity
2. Superposition

EXAMPLE: Uniaxial Stress



$$[\sigma\ ] = \begin{vmatrix} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

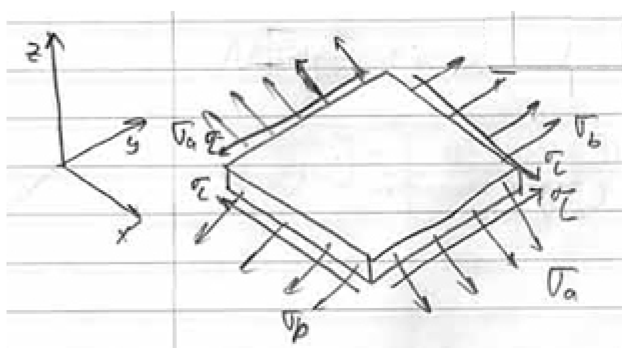
EXAMPLE: Hydrostatic Stress



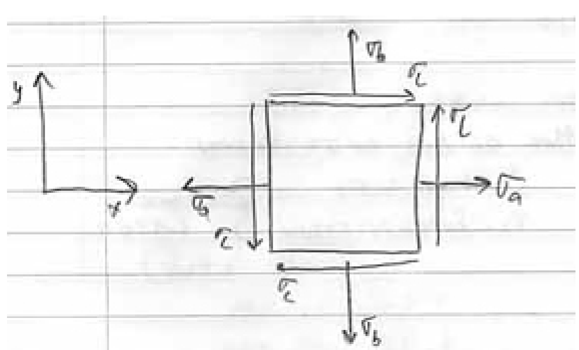
$$[\sigma\ ] = \begin{vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{vmatrix}.$$

Note: p is negative (compressive stresses are negative by convention)

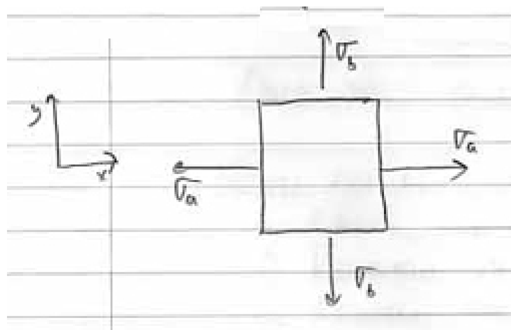
EXAMPLE: Plane Stress



$$[\sigma\ ] = \begin{vmatrix} \sigma_a & \tau & 0 \\ \tau & \sigma_b & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

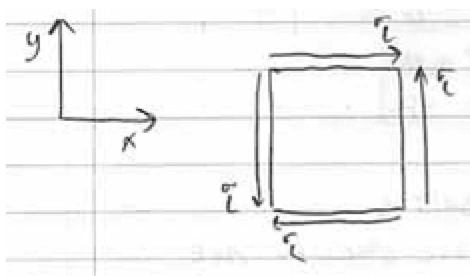


EXAMPLE: Biaxial Plane Stress



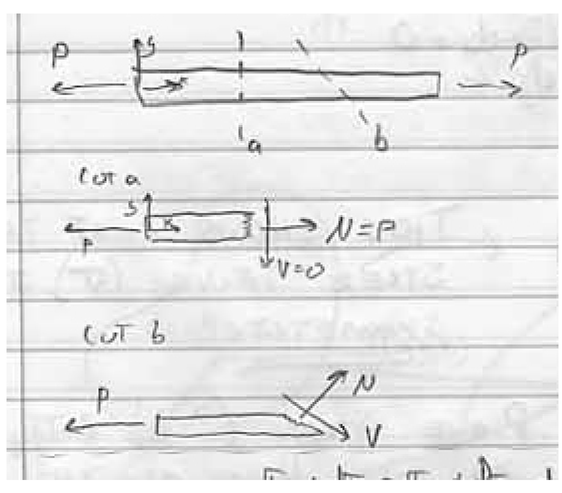
$$[\sigma\ ] = \begin{vmatrix} \sigma_a & 0 & 0 \\ 0 & \sigma_b & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

EXAMPLE: Shear Plane Stress

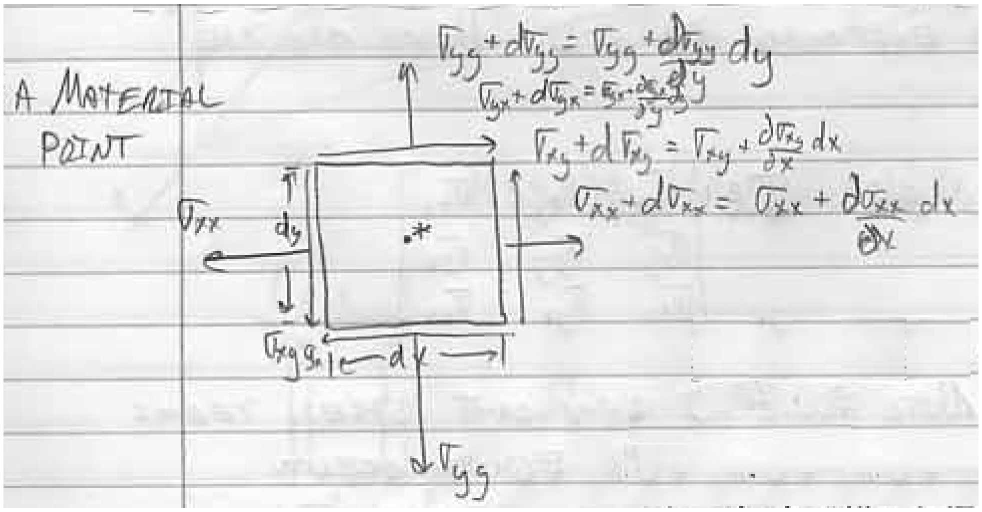


$$[\sigma\ ] = \begin{vmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Note: The stress at a material point that you see, and its magnitude, depends on the orientation of the coordinates relative to your loading.
In other words: the state of stress is a function of the chosen coordinate system.



This is called a ”stress transformation”



Equilibrium:

$$\sum F_x = 0$$

$$-\sigma_{xx} dydz + (\sigma_x x + \frac{\partial \sigma_{xx}}{\partial x} dx) dydz - \sigma_{yx} dxdz + (\sigma_{yx} \frac{\partial \sigma y x}{\partial y} dxdydz = 0$$

So:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}$$

$$\sum F_y = 0$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x}$$

$$\sum M$$

$$(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x} dx) \frac{dx}{2} dydz + \sigma_{xy} dydz \frac{dx}{2} - \sigma_{yx} dxdz \frac{dy}{2} - (\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy) dxdz \frac{dy}{2} = 0$$

$$\sigma_{xy} - \sigma_{yx} + \frac{\partial \sigma_{xy}}{\partial x} \frac{dx}{2} - \frac{\partial \sigma_{yx}}{\partial y} \frac{dy}{2} = 0$$

For $dx, dy \to 0$:

$$\sigma_{xy} = \sigma_{yx}$$

This shows that the stress tensor (σ) is *symmetric*.

This was derived in plane stress (2-D) but it can be extended to 3-D. Here are the results:

$$\sigma_{xy} = \sigma_{yx}$$

$$\sigma_{yz} = \sigma_{zy}$$

$$\sigma_{xz} = \sigma_{zx}$$

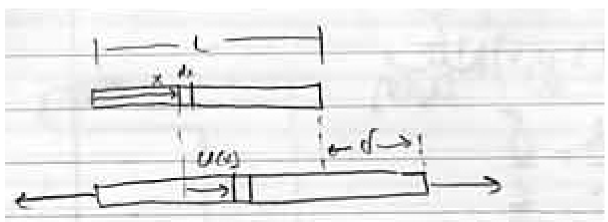
So:

$$[\sigma\ ] = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix}.$$

Note: 6 independent stress terms due to equilibrium. The stress tensor is symmetric.

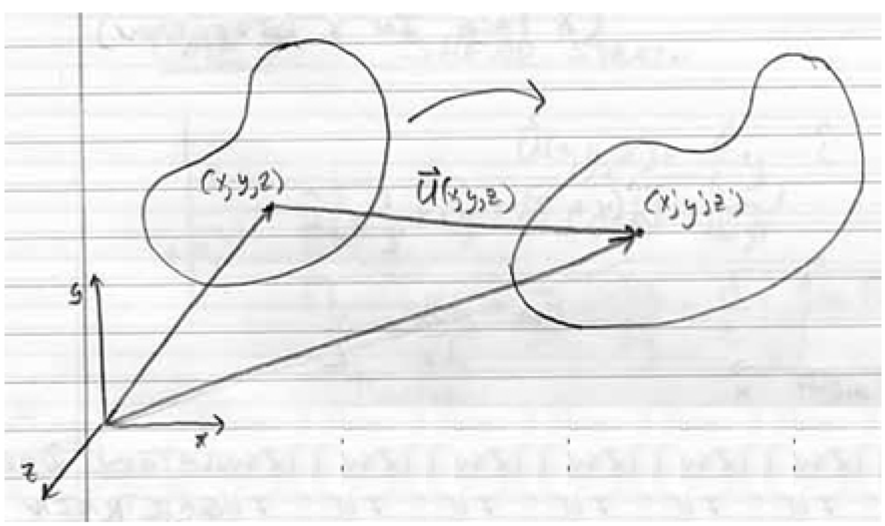
Strain:

Uniaxial:



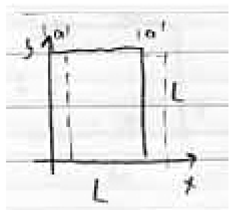
$$\epsilon(x) = \frac{du(x)}{dx}$$

$$\epsilon = \frac{\delta}{L}$$



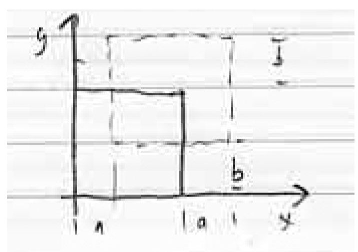
$$\vec{u}(x,y,z) = u_x(x,y,z)\hat{i} + u_y(x,y,z)\hat{j} + u_z(x,y,z)\hat{k}$$

Case 1:



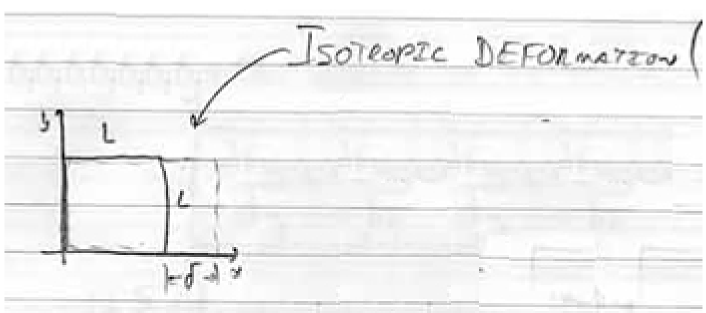
$$\vec{u}(x,y,z) = a\hat{i}$$

No strain. Rigid body translation.
Case 2:



$$\vec{u}9x,y,z) = a\hat{i} + b\hat{j}$$

No strain. Rigid body translation.
Case 3:



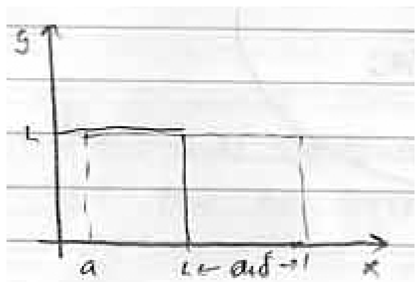
Isotropic Deformation: Stentches evenly so this is linear.

$$\vec{u}(x, y, z) = \frac{\delta x}{L}\hat{i}$$
$$\frac{du_x}{dx} = \frac{\delta}{L}$$

Define:

$$\epsilon_{xx} = \frac{du_x}{dx}, \text{ Normal Strain (x face in x direction)}$$

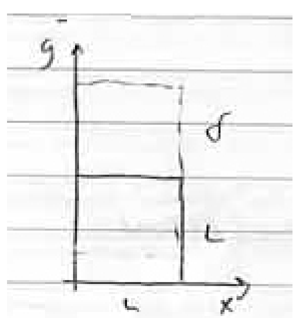
Case 4:



$$\vec{u}(x, y, z) = \left(a + \frac{\delta}{L}x\right)\hat{i}$$
$$\epsilon_{xx} = \frac{du_x}{dx} = \frac{\delta}{L}$$

Note: Rigid body translation does not effect the strain.

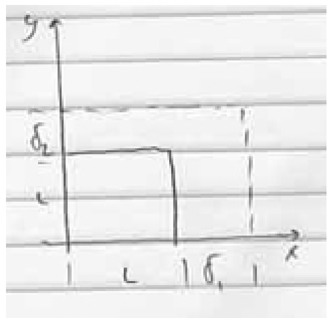
Case 5:



$$\vec{u}(x,y,z) = \frac{\delta}{L} y \hat{j}$$

$$\epsilon_{yy} = \frac{du_y}{dy} = \frac{\delta}{L}$$

Case 6:

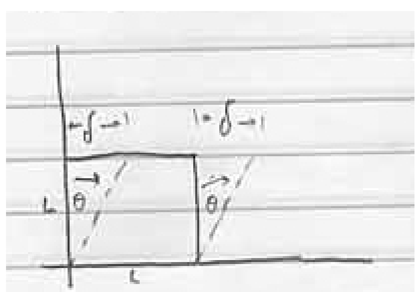


$$\vec{u}(x,y,z) = \frac{\delta_1}{L} x \hat{i} + \frac{\delta_2}{L} y \hat{j}$$

$$\epsilon_{xx} = \frac{du_x}{dx} = \frac{\delta_1}{L}$$

$$\epsilon_{yy} = \frac{du_y}{dy} = \frac{\delta_2}{L}$$

Case 7: Shear Strain

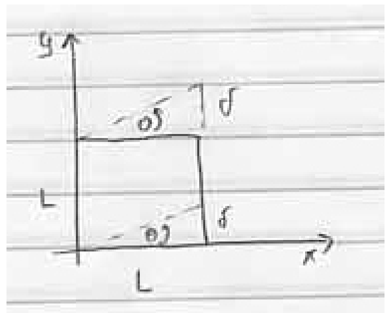


$$\vec{u}(x,y,z) = \frac{\delta}{L} y \hat{j}$$

$$\frac{\partial u_x}{\partial y} = \frac{\delta}{L} = \tan\theta \approx \theta$$

Define:

$$\gamma_{xy} = \theta$$



$$\vec{u}(x,y,z) = \frac{\delta}{L} x\hat{j}$$

$$\frac{\partial u_y(x)}{\partial x} = \frac{\delta}{L} = \theta$$

$$\gamma_{yx} = \theta$$

$$\gamma_{yx} = \frac{du_x}{dy} + \frac{du_y}{dx}$$

Define:

$$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right]$$