NONLINEAR MECHANICAL SYSTEMS

CANONICAL TRANSFORMATION S AND NUMERICAL INTEGRATION

Jacobi Canonical Transformations

A Jacobi canonical transformations yields a Hamiltonian that depends on only one of the conjugate variable sets.

Assume dependence on new momentum alone.

$H(p^*,q^*) = K(p^*)$

$\partial K(p^*)/\partial q^* = 0$

Thus

$dp^*/dt = e^*$

$dq^*/dt = \partial K(p^*)/\partial p^* - f^*$

The simple relation between effort and the rate of change of momentum is recovered in the new coordinates.

EXAMPLE: SIMPLE HARMONIC OSCILLATOR

Hamiltonian

$$H(p,q) = \frac{1}{2}(p^2/I + q^2/C)$$

Hamilton's equations

$$dq/dt = \partial H/\partial p = p/I$$

$$dp/dt = -\partial H/\partial q = q/C$$

Change variables from old (q,p) to new (P,Q)

Define $Z_o = \sqrt{IC}$ and the generating function

$$S(q,Q) = Z_o(q^2/2) \cot Q$$

The transformation equations are

$$p = \partial S/\partial q = Z_o q \cot Q$$

$$P = -\partial S/\partial Q = Z_o(q^2/2)/\sin^2 Q$$

Express the old variables in terms of the new

$p = \sqrt{2P}\ \cos Q \sqrt{Z_o}$

$q = \sqrt{2P}\ \sin Q\ (1/\sqrt{Z_o}\ )$

Define $\omega_o = \sqrt{1/IC}$ and the new Hamiltonian is

$H(P,Q) = \omega_o P = K(P)$

Hamilton's equations in the new coordinates

$dQ/dt = \partial K/\partial P = \omega_o$

$dP/dt = -\partial K/\partial Q = 0$

Their solution is

$Q(t) = \omega_o t + \text{constant}$

$P(t) = \text{constant}$

In essence this variable change has integrated the equations.

As the product of P and Q has the units of action (energy by time) it is sometimes called a (simple harmonic) actional transformation.

PHYSICAL INTERPRETATION:

P is proportional to the total system energy.

Its square root is proportional to oscillation amplitude.

Q is the phase angle of the oscillations.

In general, finding Jacobi canonical transformations requires solving a non-trivial partial differential equation.

A practical alternative is to separate the Hamiltonian into two parts, one with a known Jacobi canonical transform.

$H(p,q) = H_j(p,q) + H_n(p,q)$

Apply the known Jacobi canonical transformation

$H^*(P,Q) = H_j^*(P) + H^*_n(P,Q)$

We may represent the second term as a set of canonical forces

$e^*(P,Q) = -\partial H^*_n/\partial Q$

$f^*(P,Q) = -\partial H^*_n/\partial P$

The transformed equations become

$dP/dt = e^*(P,Q)$

$dQ/dt = \partial H^*_j/\partial P - f^*(P,Q)$

An advantage of this change of variables is that, in effect, it integrates the fundamental oscillatory mode of the solution.

EXAMPLE: SIMPLE PENDULUM

For large amplitudes, the simple pendulum is a nonlinear oscillator.

$H(\eta,\theta) = \eta^2/2 + 1 - \cos\theta$

where

θ angle with respect to the vertical

η corresponding angular momentum

Expand the cosine as a power series

$H(\eta,\theta) = \eta^2/2 + \theta^2/2 - \theta^4/4! + \theta^6/6! - \ldots$

The Hamiltonian is quadratic in momentum and displacement with additional terms in displacement of fourth power and higher.

Until the fourth power of the angle in radians becomes significant,

the nonlinear pendulum may be treated as linear system

with a Hamiltonian that is quadratic in momentum and displacement.

For the quadratic terms have a knownJacobi canonical transformation: the simple harmonic actional.

Split the Hamiltonian as follows

$H(\eta,\theta) = \eta^2/2 + \theta^2/2 + (1 - \cos\theta - \theta^2/2)$

$H(\eta,\theta) = K(\eta,\theta) + N(\eta,\theta)$

Apply the simple harmonic actional

$\theta = \sqrt{2P}\ \sin Q$

$\eta = \sqrt{2P}\ \cos Q$

The Hamiltonian becomes

$H^*(P,Q) = K^*(P) + N^*(P,Q)$

In the original variables, the system equations are

$d\eta/dt = -\partial H/\partial\theta$

$d\theta/dt = \partial H/\partial\eta$

In the new variables, the system equations become

$dP/dt = -\partial N^*/\partial Q$

$dQ/dt = 1 + \partial N^*/\partial P$

Transformation does not change the value of either K or N.

Use the chain rule on the original N which depends only on θ.

$\partial N^*/\partial Q = (\partial N/\partial \theta)\ (\partial \theta/\partial Q)$

$\partial N^*/\partial P = (\partial N/\partial \theta)\ (\partial \theta/\partial P)$

$\partial N/\partial \theta = \sin\theta - \theta$

$\partial \theta/\partial Q = \sqrt{2P}\ \cos Q$

$\partial \theta/\partial P = \sin Q\ (1/\sqrt{2P}\ )$

The transformed equations become

$dP/dt = [\sqrt{2P}\ \sin Q - \sin(\sqrt{2P}\ \sin Q)][\sqrt{2P}\ \cos Q]$

$dQ/dt = 1 + [\sin(\sqrt{2P}\ \sin Q) - \sqrt{2P}\ \sin Q][\sin Q\ (1/\sqrt{2P}\ )]$

Use the transformation equations to express the rates of change as a function of both old and new variables.

$dP/dt = (\theta - \sin\theta)\eta$

$dQ/dt = 1 + (\sin\theta - \theta)\theta/2P$

$\theta = \sqrt{2P}\ \sin Q$

$\eta = \sqrt{2P}\ \cos Q$

What have we gained?

The system equations are simpler in the old variables

$d\eta/dt = -\sin\theta$

$d\theta/dt = \eta$

In the new variables, the solution is far more stable numerically.

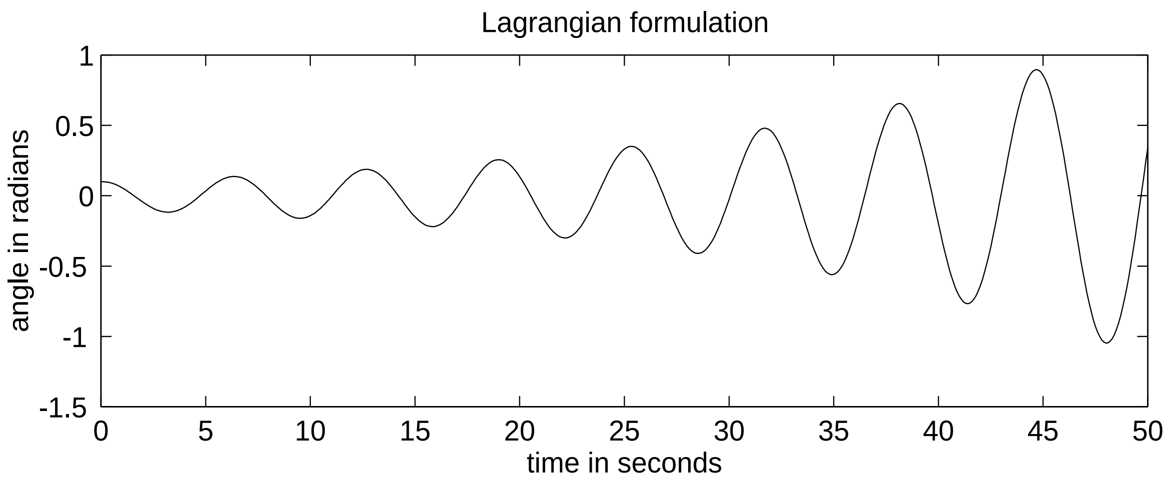
Simple Euler integration algorithm

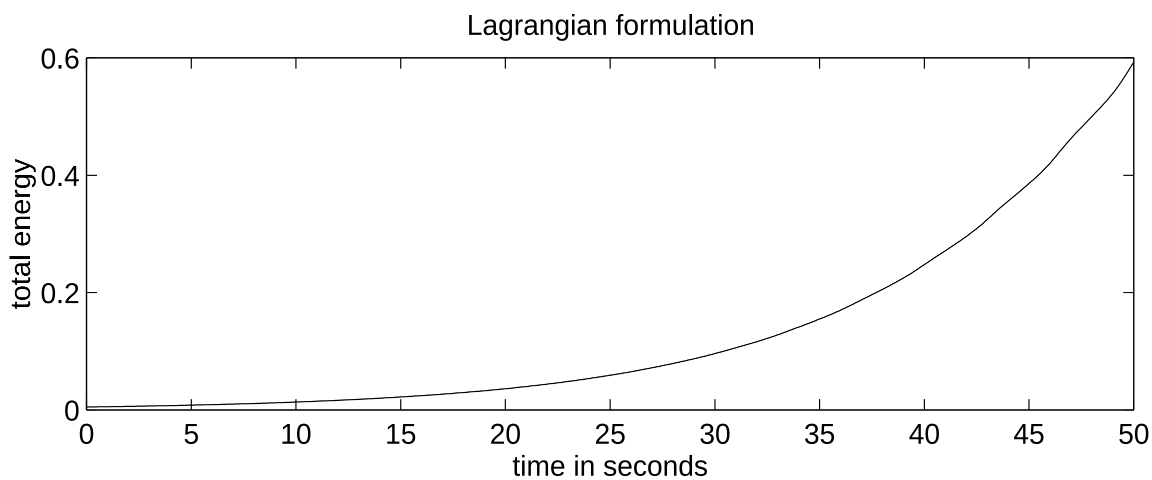
starting time 0 seconds, final time 50 seconds, time step 0.1 seconds.

start from rest at an angle of 0.1 radians (≈6°)

In old coordinates, simulation is unstable.

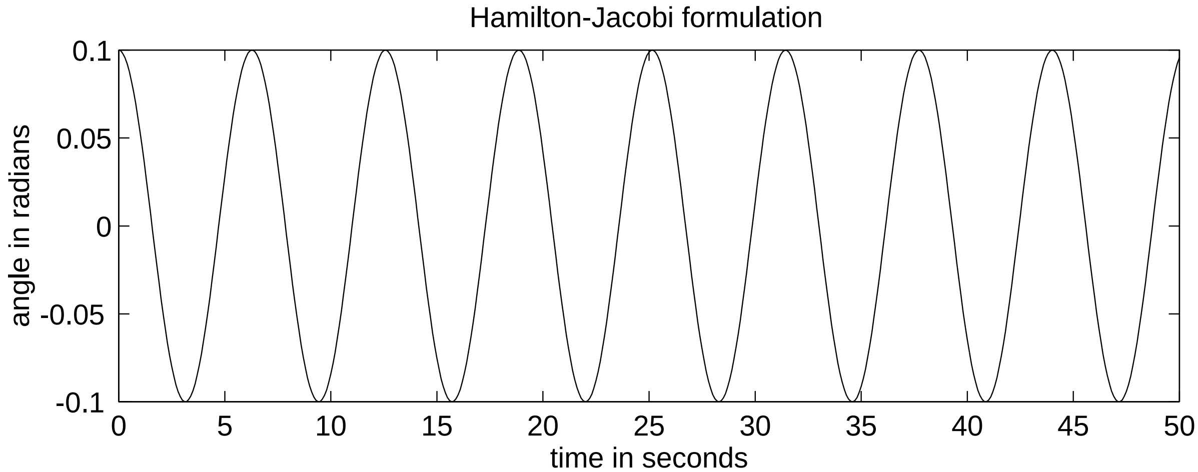
Total system energy grows exponentially.

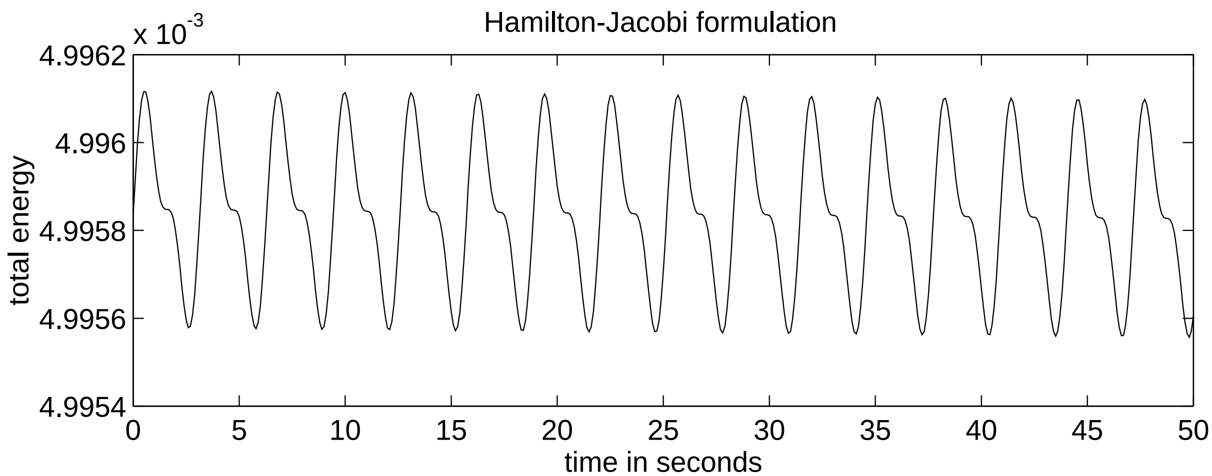




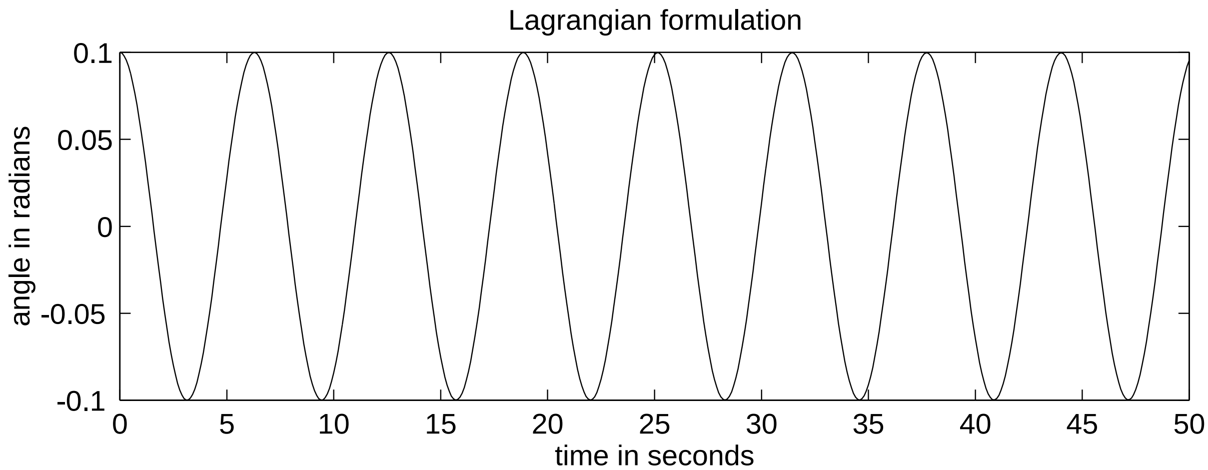
In new coodinates, the simulation is stable.

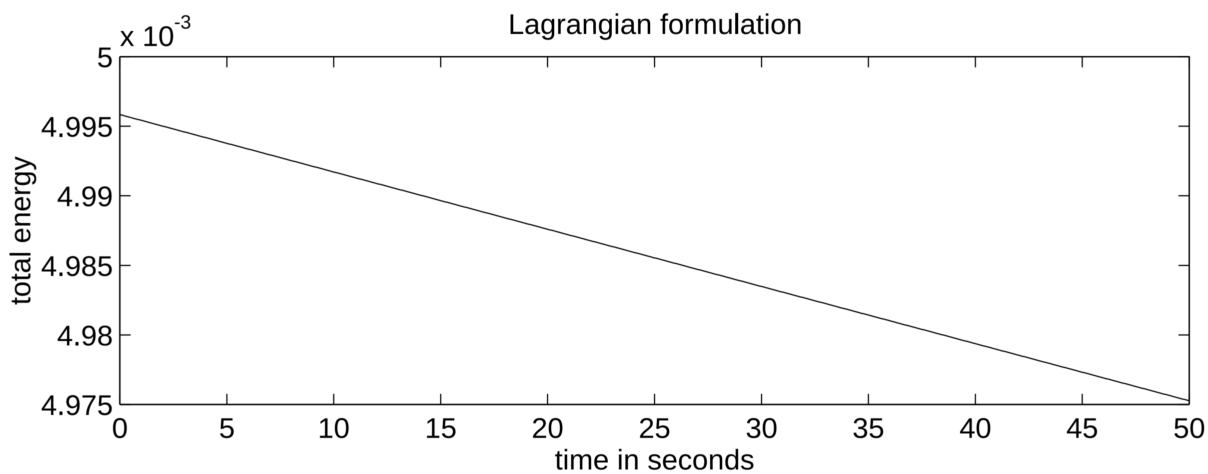
Total system energy variation: $5.6x10^{-7}$.



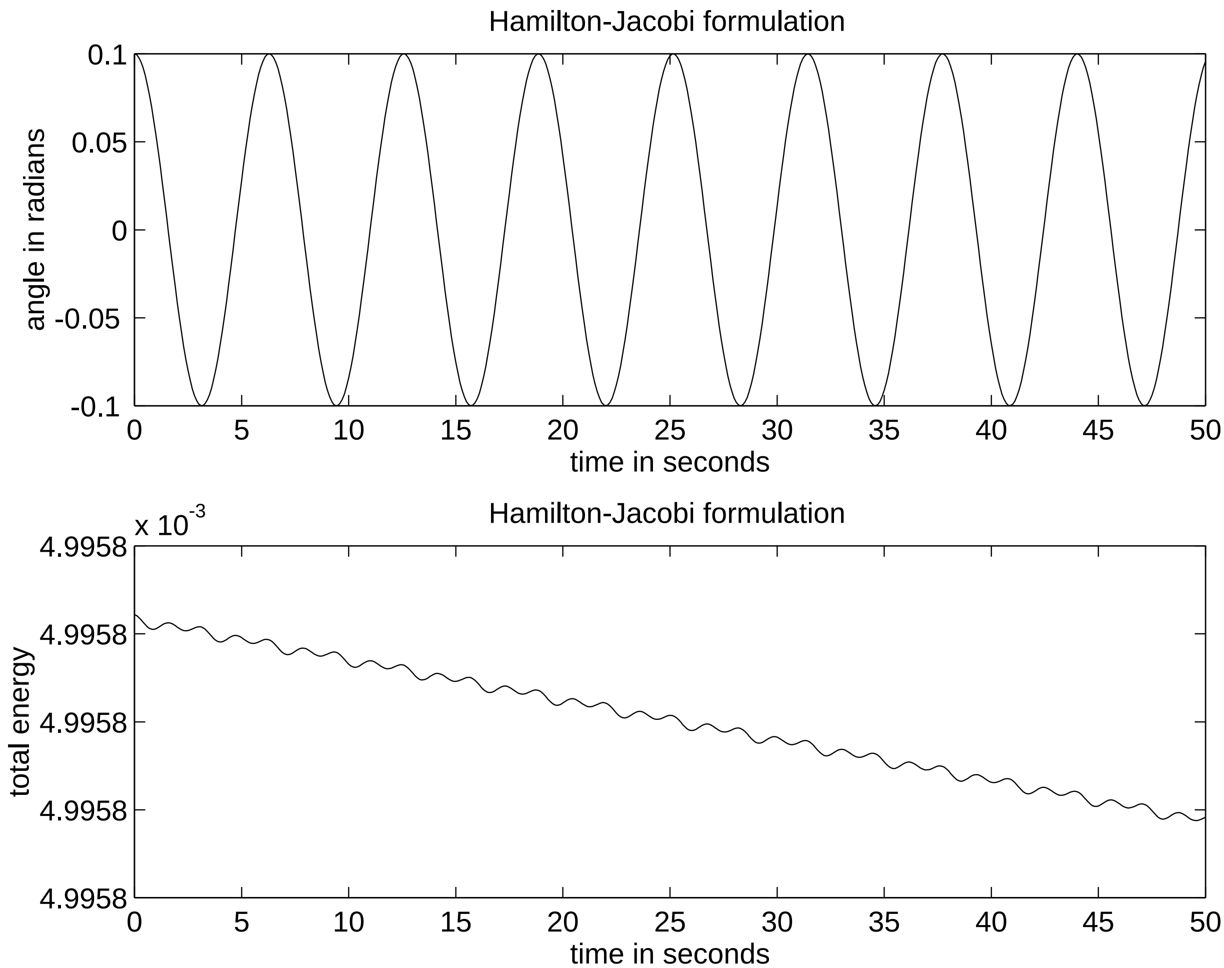


Perform the same integration using a third-order fixed-step Runge-Kutta algorithm.



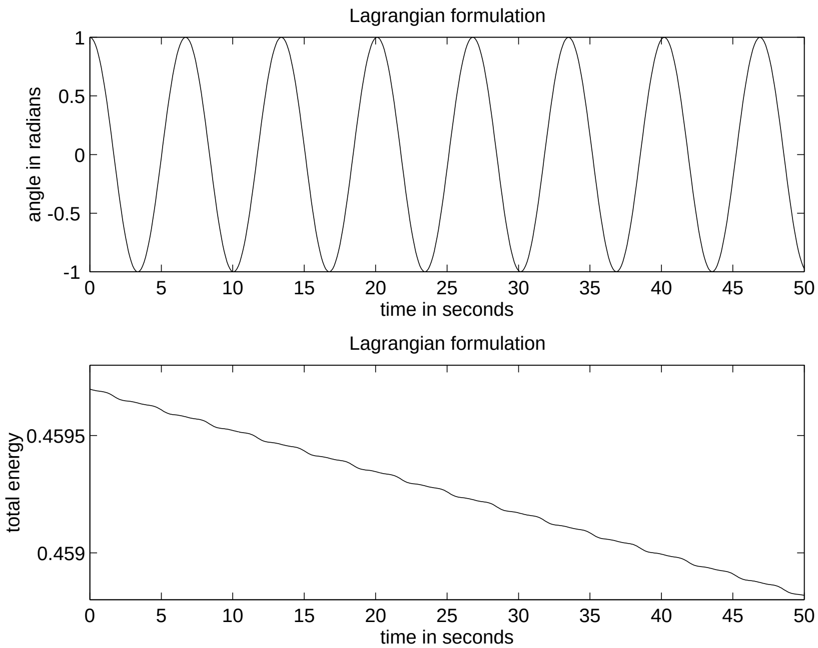


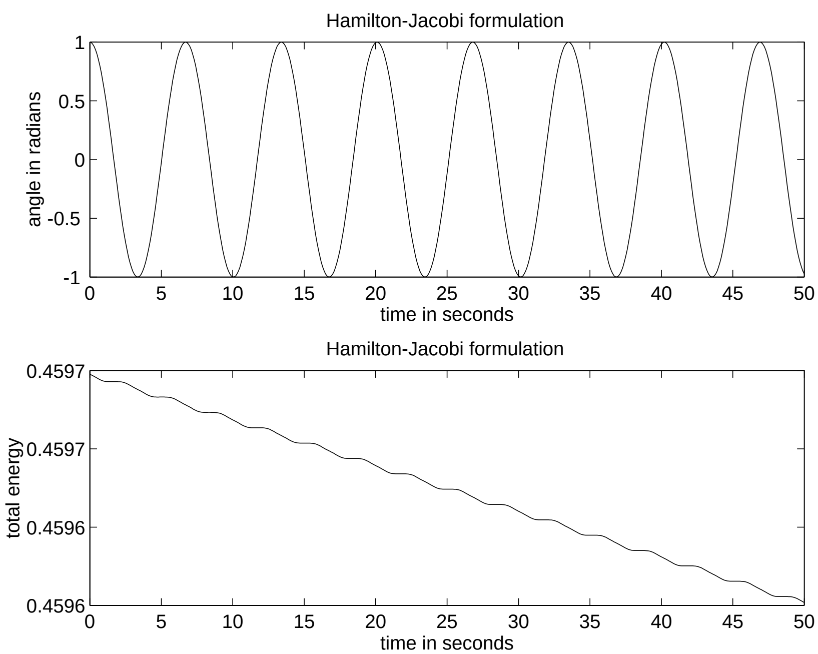
Total energy, declines steadily by 2.1×10^{-5} over 50 seconds.



Total energy also decreases, but by 2.3×10^{-10} — a hundred thousand time less.

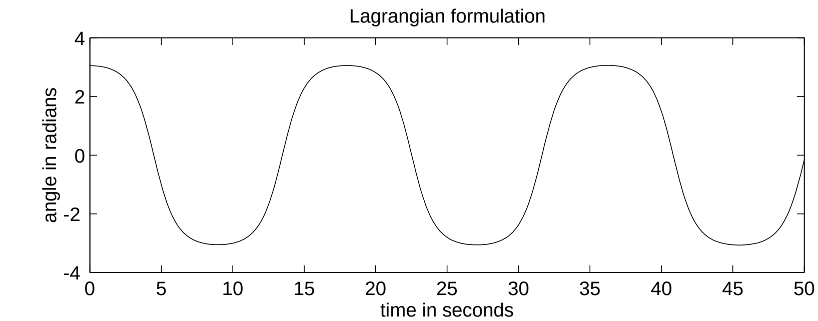
Start the pendulum from rest at 1 radian (≈57°) and use the same integration algorithm and parameters

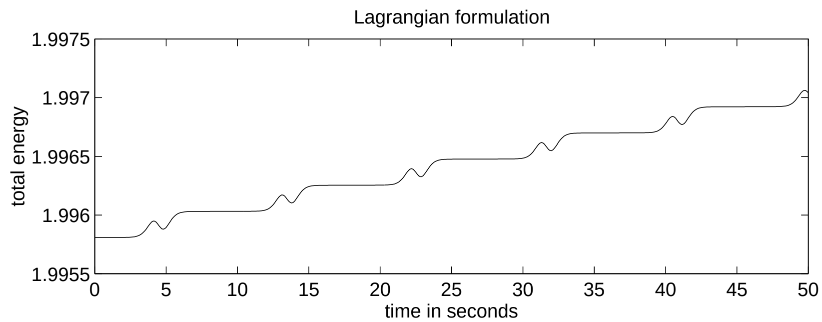


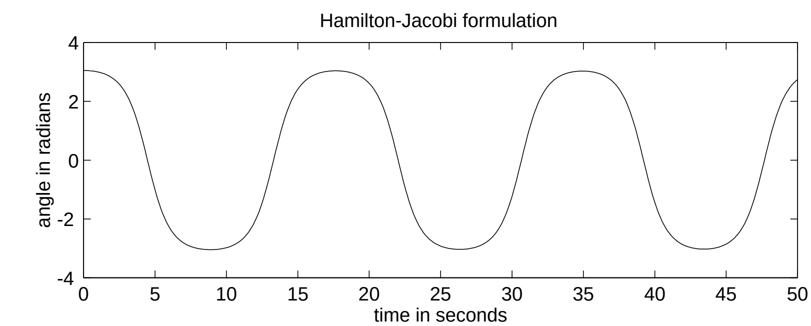


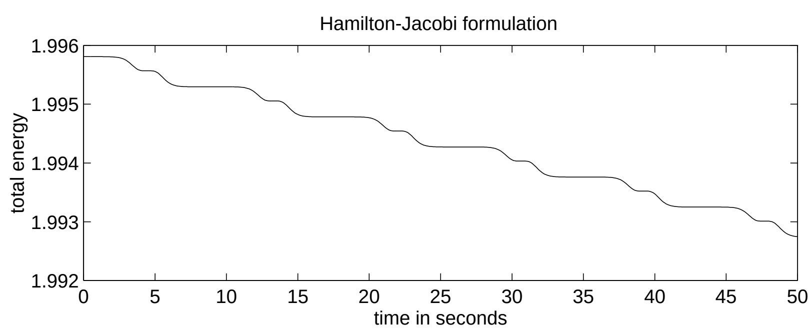
Again, the transformed equations produce a smaller decline in energy, though the difference is less pronounced — 8.8×10^{-4} vs. 1.5×10^{-4}.

Start from rest at 5° off vertically upright (3.05 radians) and use the same integration algorithm and parameters



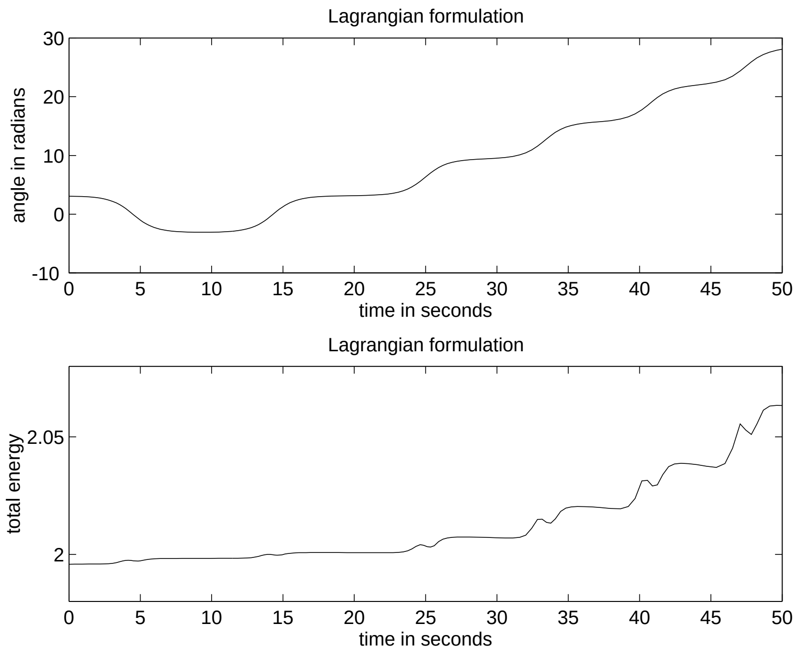






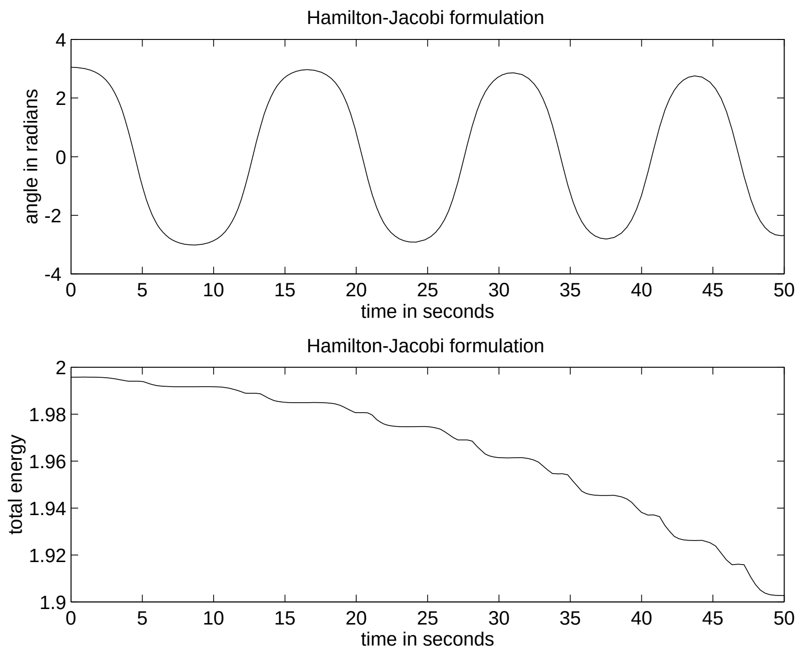
Now the original formulation is unstable — energy increases by $1.3x10^{-3}$ in 50 seconds. The transformed equations yield a decline of energy of $3.1x10^{-3}$.

Start from the same initial conditions but use MATLAB's ode23, a 3rd-order adaptive Runge-Kutta algorithm with error tolerance of 1.0×10^{-3}



Now the steady increase in computed total energy in the original formulation results in a major departure of the computed angle from what it should be

— the simulation claims that after one oscillation the pendulum will spin continuously in one direction.



The transformed equations do not exhibit this behavior, though the computed energy declines substantially (9.3×10^{-2} in 50 seconds).

POINTS:

- **Never believe anything you get from a computer. Find some way of cross checking the results. One effective method is to compute a known invariant, in this case energy.**

- **The equations in the original variables may look simpler, but that is deceptive. In fact the transformed equations have been partially integrated by the transformation and so present a less demanding task to the integration algorithm.**

- **A little analysis up front can have a dramatic effect on the accuracy of numerical computations.**