Problem 2 – Solution

(a) The 4 equations are those we wrote in recitation, just expressed in terms of z. That is

(1) Conservation of momentum: $\frac{\partial \sigma_{zz}^{tot}}{\partial z} = 0$

(2) Darcy's law: $U_z = -k\frac{\partial p}{\partial z}$

(3) Constitutive law: $\sigma_{zz}^{tot} = (2G+\lambda)\varepsilon_{zz} - p$

where $H = 2G+\lambda$

(4) Mass conservation: $U_z = -\frac{\partial u_z}{\partial t} + U_{z0}$

where U_{z0} is the area-averaged velocity at a point where the solid is not moving, $= U_0$

(b) Combining (1)-(4) we obtain, for the <u>general</u> case:

$$\frac{\partial \sigma_{zz}^{tot}}{\partial z} = 0 = H\frac{\partial \varepsilon_{zz}}{\partial z} - \frac{\partial p}{\partial z}$$

And $$H\frac{\partial \varepsilon_{zz}}{\partial z} = H\frac{\partial^2 u_z}{\partial z^2}$$ [from definition of strain]

$$\frac{\partial p}{\partial z} = \frac{U_z}{k}$$ [from (2)]

$$= \frac{1}{k}\left(\frac{\partial u_z}{\partial t} - U_{z0}\right)$$ [from (4)]

Or $$\frac{\partial u_z}{\partial t} - U_{z0} = Hk\frac{\partial^2 u_z}{\partial z^2}$$

Note that the term U_{z0} is only = 0 in cases for which there is one non-porous boundary (as in the handout from class) but not in general.

- This is where we went wrong in recitation!

(c) Now we can assume steadiness ($\partial/\partial t = 0$) and obtain

$$-U_{z0} = -U_0 = Hk\frac{\partial^2 u_z}{\partial z^2} \quad (1)$$

Or $$-\frac{U_0}{Hk} = \frac{\partial^2 u_z}{\partial z^2} \quad (2)$$

Multiply by dz and integrate once:

$$-\frac{U_0}{Hk}z + c_1 = \frac{\partial u_z}{\partial z} \quad (3)$$

And integrate again:

$$-\frac{U_0}{Hk}\frac{z^2}{2} + c_1 z + c_2 = u_z \quad (4)$$

Boundary conditions are that:

$$u_z(z=0)=0 \quad \Rightarrow \quad c_2=0$$

And $$\sigma_{zz}^{tot}(z=-L)=-p_0 \qquad \therefore \quad \varepsilon_{zz}=\left.\frac{du_z}{dz}\right|_{z=-L}=0$$

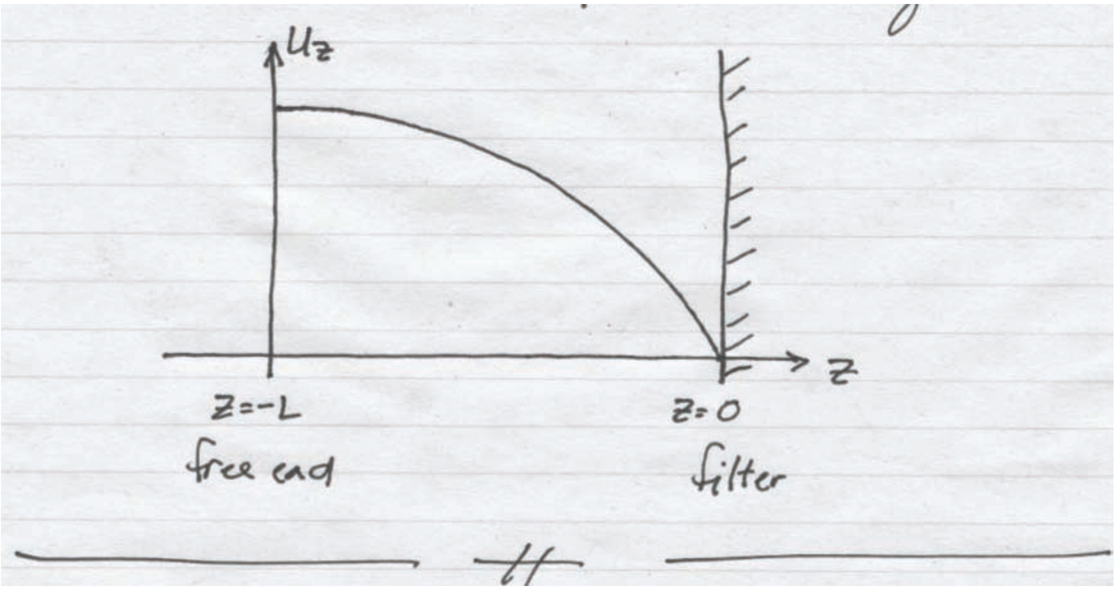
Using (3):

$$-\frac{U_0}{Hk}(-L)+c_1=0$$

$$c_1=-\frac{U_0 L}{Hk}$$

So $$u_z(z)=-\frac{U_0}{Hk}\left(\frac{z^2}{2}+Lz\right)$$

This solution satisfies the boundary conditions we discussed in recitation, and has the form



The problem in recitation arose from starting with the equation

$$\frac{\partial u_1}{\partial t}=Hk\frac{\partial^2 u_1}{\partial x_1^2}$$

rather than the more general expression in (b) above!!!