Suggested reading: Strogatz, Chapter 9.

PROBLEM SET 7

We have provided the program, `lorenz.m`, to compute time series for the Lorenz model. You will also need to compute the time-dependent separation between points in two time series for Question 6. (See the supplement for hints on how to do this with `Matlab`.)

The Lorenz model is given by

$$\begin{aligned}\dot{X} &= PY - PX \\ \dot{Y} &= -XZ + rX - Y \\ \dot{Z} &= XY - bZ\end{aligned}$$

where P is the Prandtl number (usually taken to be equal to 10), b is a constant (usually taken to equal 8/3) and r is the ratio of the Rayleigh number to the critical Rayleigh number.

1. Find the location (analytically) of the three steady state solutions for the Lorenz model. Give a physical interpretation of these steady state solutions.
2. Show analytically that the solution corresponding to conduction becomes unstable for $r > 1$.
3. Show analytically that the steady-state solutions corresponding to convection become unstable for
$$r > r_c = \frac{P(P+3+b)}{P-1-b}$$
4. Verify *numerically* (with plots) that steady-state convection is unstable for $r > r_c$ (use P=10, b=8/3). $r = 28$ gives nice results. Give a physical interpretation of the time dependent behaviour you have found for $X(t)$, $Y(t)$, and $Z(t)$. You may wish to make Poincaré sections (e.g., projections in the XY, XZ, and YZ planes).
5. Find $X(t)$ for $r = 166$ and $r = 166.1$. What observations can you make? Remember to let the transients die out.
6. Verify exponential divergence of small differences in initial conditions for $r = 170$. Assuming $\delta(t) = \delta_0 e^{\lambda t}$, where $\delta(t)$ is the distance between two points in phase space, find the value of λ, the largest Lyapanov exponent. What accounts for the long-time behaviour of $\delta(t)$?
7. Obtain another example of your own choosing of some interesting behaviour of the Lorenz model and describe what you found.