17 Intermittency (and quasiperiodicity)

In this lecture we discuss the other two generic routes to chaos, intermittency and quasiperiodicity.

Almost all our remarks will be on intermittency; we close with a brief description of quasiperiodicity.

Definition: Intermittency is the occurrence of a signal that alternates randomly between regular (laminar) phases and relatively short irregular bursts.

In the exercises we have already seen examples, particulary in the Lorenz model (where it was discovered, by Manneville and Pomeau).

Examples:

- The Lorenz model, near $r = 166$.
  Figure 1a,b Manneville and Pomeau (1980)
- Rayleigh-Benard convection.
  BPV, Figure IX.9

17.1 General characteristics of intermittency

Let r = control parameter. The following summarizes the behavior with respect to r:

- For $r < r_i$, system displays stable oscillations (e.g., a limit cycle).
- For $r > r_i$ ($r - r_i$ small), system in in the *intermittent* regime: stable oscillations are interrupted by fluctuations.
- As $r \to r_i$ from above, the fluctuations become increasingly rare, and disappear for $r < r_i$.
- Only the *average intermission time* between fluctuations varies, not their amplitude nor their duration.

We seek theories for

- Linear stability of the limit cycle and "relaminarization." (i.e. return to stability after irregular bursts).
- Scaling law for intermission times.
- Scaling law for Lyaponov exponents.

17.2 One-dimensional map

We consider the instability of a Poincaré map due to the crossing of the unit circle at (+1) by an eigenvalue of the Floquet matrix.

This corresponds to the specific case of *Type I intermittency.*

Let u be the coordinate in the plane of the Poincaré section that points in the direction of the eigenvector whose eigenvalue λ crosses +1.

The lowest-order approximation of the 1-D map constructed along this line is

$$u' = \lambda(r)u. \tag{39}$$

Taking $\lambda(r_i) = 1$ at the intermittency threshold, we have

$$u' = \lambda(r_i)u = u. \tag{40}$$

We consider this to be the leading term of a Taylor series expansion of $u'(u, r)$ in the neighborhood of $u = 0$ and $r = r_i$.

Expand to first order in $(r - r_i)$ and second order in u:

$$u'(u,r) \simeq u'(0,r_i) + u \cdot \left.\frac{\partial u'}{\partial u}\right|_{0,r_i} + \frac{1}{2}u^2 \cdot \left.\frac{\partial^2 u'}{\partial u^2}\right|_{0,r_i} + (r - r_i) \left.\frac{\partial u'}{\partial r}\right|_{0,r_i}$$

Evaluating equation (39), we find that the first term vanishes:

$$u'(u = 0, r = r_i) = 0.$$

From equation (40), we have

$$\left.\frac{\partial u'}{\partial u}\right|_{0,r_i} = \lambda(r_i) = 1.$$

Finally, rescale u such that

$$\frac{1}{2}\left.\frac{\partial^2 u'}{\partial u^2}\right|_{0,r_i} = 1$$

and set

$$\varepsilon \propto (r - r_i).$$

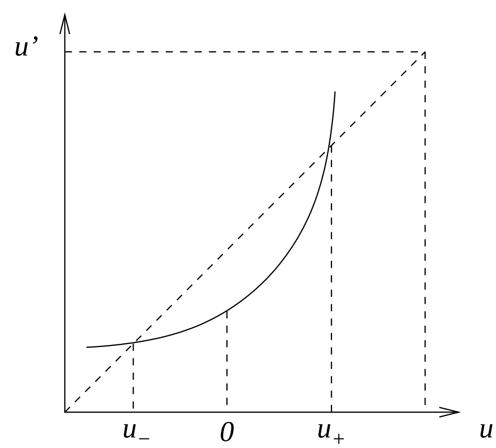
The model now reads

$$u' = u + \varepsilon + u^2,$$

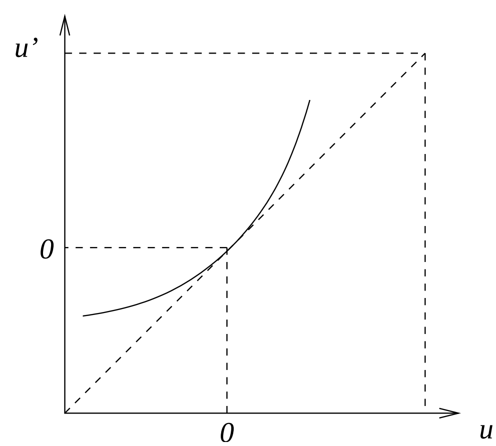
where ε is now the control parameter.

Graphically, we have the following system:

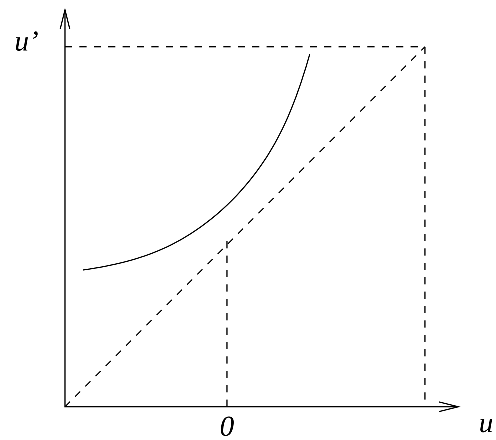
- $\varepsilon < 0$, i.e. $r < r_i$.
- u_- is stable fixed point.
- u_+ is unstable.



- $\varepsilon = 0$, i.e. $r = r_i$.
- u' is tangent to indentity map.
- $u_- = u_+ = 0$ is marginally stable.

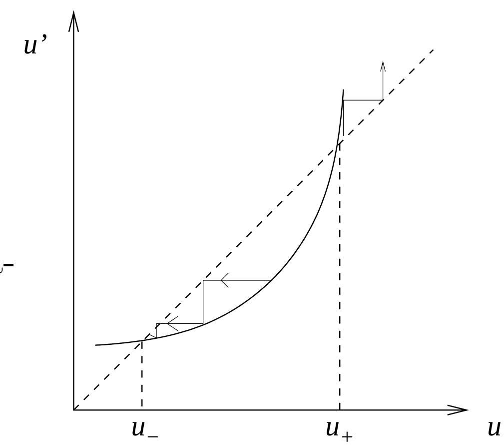


- $\varepsilon > 0$, i.e. $r > r_i$.
- no fixed points.



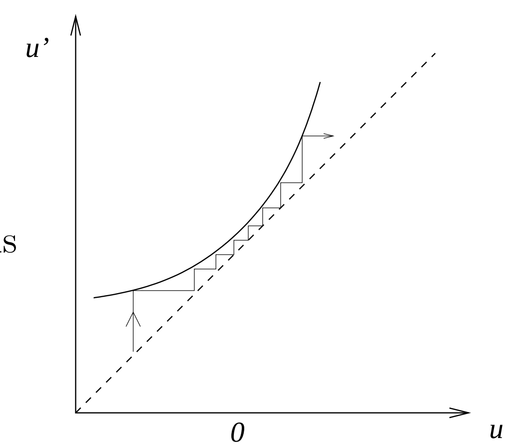
For $\varepsilon < 0$, the iterations look like

- u_- is an attractor for initial conditions $u < u_+$.
- For initial conditions $u > u_+$, the iterations diverge.



The situation changes for $\varepsilon > 0$, i.e. $r > r_i$:

- No fixed points.
- Iterations beginning at $u < 0$ drift towards $u > 0$.



The fixed points of $u'(u)$ represent stable oscillations of the continuous flow.

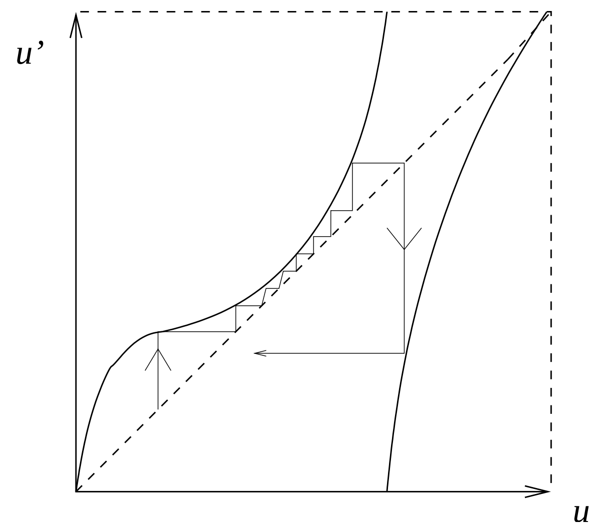
Thus for $u \simeq 0$, the drift for $\varepsilon > 0$ corresponds to a flow qualitatively similar to the stable oscillations near $u = 0$ for $\varepsilon < 0$.

However, when $\varepsilon > 0$, there is no fixed point, and thus no periodic solution.

The iterations eventually run away and become unstable—this is the *intermittent* burst of noise.

How does the laminar phase begin again, or "relaminarize"?

Qualitatively, the picture can look like



Note that the precise timing of the turbulent burst is unpredictable.

The discontinuity is *not* inconsistent with the presumed continuity of the underlying equations of motion—this is a map, not a flow.

Moreover the Lorenz map itself contains a discontinuity, corresponding to the location of the unstable fixed point.

17.3 Average duration of laminar phase

What can we say about the average duration of the laminar phases?

Writing our theoretical model as a map indexed by k, we have

$$u_{k+1} = u_k + \varepsilon + u_k^2.$$

For $u_{k+1} \simeq u_k$, we can instead write the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}k} = \varepsilon + u^2.$$

The general solution of this o.d.e. is

$$u(k) = \varepsilon^{1/2} \tan \left[\varepsilon^{1/2}(k - k_0) \right].$$

Take $k_0 = 0$, the step at which iterations traverse the narrowest part of the channel.

We thus have

$$u(k) = \varepsilon^{1/2} \tan \left(\varepsilon^{1/2} k \right).$$

We see that $u(k)$ diverges when

$$\varepsilon^{1/2} k = \pm \frac{\pi}{2} \quad \text{or} \quad k = \pm \frac{\pi}{2} \varepsilon^{-1/2}.$$

The divergence signifies a turbulent burst.

When $k \sim \varepsilon^{-1/2}$, $u_{k+1} - u_k$ is no longer small, and the differential approximation of the difference equation is no longer valid.

Thus: if $\tau =$ time ($\propto$ number of iterations) needed to traverse the channel, then

$$\tau \propto \varepsilon^{-1/2} \quad \text{or} \quad \tau \propto (r - r_i)^{-1/2}. \tag{41}$$

Thus the laminar phase lasts increasingly long as the threshold $r = r_i$ is approached from above.

17.4 Lyaponov number

We can also predict a scaling law for the Lyaponov number.

Near the fixed point ($u \simeq 0$, $\varepsilon > 0$), the increment δu_{k+1} due to an increment u_k is, to first order,

$$\delta u_{k+1} \simeq \lambda_1 \delta u_k$$

where λ_1 is eigenvalue that passes through (+1).

After N iterations,

$$\delta u_N \simeq \lambda_N \lambda_{N-1} \lambda_{N-2} \cdots \lambda_1 \delta u_1.$$

Suppose $N \simeq$ the duration of the laminar phase. Then

$$\lambda_N > 1 \quad \text{and} \quad \lambda_{N-1} \simeq \lambda_{N-2} \simeq \cdots \simeq \lambda_1 \simeq 1.$$

The Lyapunov number Λ is

$$\Lambda = \frac{1}{N} \prod_i \lambda_i \simeq \frac{\lambda_N}{N} \propto \frac{1}{N} \propto \frac{1}{\tau} \propto \sqrt{\varepsilon}.$$

where the last relation used equation (41). (Recall that $\ln \Lambda$ = Lyaponov *exponent*.)

Results from the Lorenz model verify this prediction. The "intermittent channel" of the Lorenz map is seen in

BPV, Figure IX.14

and the associated $\varepsilon^{1/2}$ scaling of the Lyaponov number is seen in

BPV, Figure IX.15

Behavior qualitatively similar to that predicted by our model has been observed in the B-Z reaction:

BPV, Figure IX.16–17

17.5 Quasiperiodicity

Finally, we make a few remarks about the third universal route to chaos, known as *quasiperiodicity.*

Recall that there are 3 generic ways in which a limit cycle on a Poincaré map may become unstable: An eigenvalue λ of the Floquet matrix (the Jacobian of the map) crosses the unit circle at

- $+1$ (as in the example of intermittency above);
- -1 (as we saw in the introduction to period doubling); and
- $\lambda = \alpha \pm i\beta$, $|\lambda| > 1$. This corresponds to the transition via *quasiperiodicity.*

As we have seen, the latter case results in the addition of a second oscillation.

This is a *Hopf bifurcation*: the transformation of a limit cycle to a quasiperiodic flow, or a torus T^2.

The route to chaos via quasiperiodicity describes how a torus T^2 (i.e., a quasiperiodic flow) can become a strange attractor.

17.5.1 An historical note

In 1944, the Russian physicist Landau proposed a theory for the transition from laminar flow to turbulence as the Reynolds number is increased.

Briefly, he envisioned the following sequence of events as Re increases beyond Re_c:

- Laminar flow (constant velocity) becomes periodic with frequency f_1 by a Hopf bifurcation.
- Period flow→quasiperiodic flow; i.e., another Hopf bifurcation. The second frequency f_2 is incommensurate with f_1.

- More incommensurate frequencies $f_3, f_4, \ldots, f_r$ appear in succession (due to more Hopf bifurcations).
- For r large, the spectrum appears continuous and the flow (on a torus T^r) is aperiodic (i.e., turbulent).

Recall that we have learned previously that, for dissipative flows,

$$\text{dimension of phase space} > \text{attractor dimension}.$$

Thus a consequence of Landau's theory is that a system must have many degrees of freedom to become chaotic.

We now know, however, from the work of Lorenz, that

- 3 degrees of freedom suffice to give rise to a chaotic flow; and
- the chaos occurs on a strange attractor, which is distinct from a torus (since trajectories diverge on the strange attractor).

17.5.2 Ruelle-Takens theory

Lorenz's observations were deduced theoretically by Ruelle and Takens in 1971.

The Ruelle-Takens theory is the *quasiperiodic* route to chaos. As a control parameter is varied, the following sequence of events can occur:

- Laminar flow $\rightarrow$ oscillation with frequency f_1.
- A second Hopf bifurcation adds a second (incommensurate) frequency f_2.
- A third Hopf bifurcation adds a third frequency f_3.
- The torus T^3 can become unstable and be replaced by a strange attractor.

The transition is demonstrated beautifully in terms of changing power spectra in the Rayleigh-Bénard experiment described by

Libchaber et al., Figure 15

Libchaber, A., Fauve, S. and C. Laroche.1983. Two-parameter study of the routes to chaos. Physica D. 7: 73-84.

Note that the Rayleigh number of the two spectra varies by less than 1%.

Such a transition can also be seen in Poincaré sections, such as the Rayleigh-Bénard experiment of

BPV, Figures VII.20, VII.21