3 Conservation of volume in phase space

We show (via the example of the pendulum) that frictionless systems *conserve* volumes (or areas) in phase space.

Conversely, we shall see, dissipative systems *contract* volumes.

Suppose we have a 3-D phase space, such that

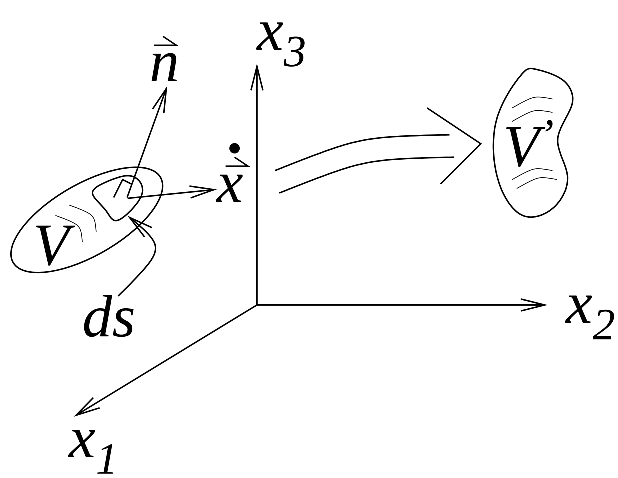
$$\begin{aligned}\dot{x_1} &= f_1(x_1, x_2, x_3)\\ \dot{x_2} &= f_2(x_1, x_2, x_3)\\ \dot{x_3} &= f_3(x_1, x_2, x_3)\end{aligned}$$

or

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x})$$

The equations describe a "flow," where $\mathrm{d}\vec{x}/\mathrm{d}t$ is the velocity.

A set of initial conditions enclosed in a volume V flows to another position in phase space, where it occupies a volume V', neither necessarily the same shape nor size:



Assume the volume V has surface S.

Let

- ρ = density of initial conditions in V;
- $\rho\vec{f}$ = rate of flow of points (trajectories emanating from initial conditions) through unit area perpendicular to the direction of flow;
- $\mathrm{d}s$ = a small region of S; and
- $\vec{n}$ = the unit normal (outward) to $\mathrm{d}s$.

Then

$$\text{net flux of points out of } S = \int_S (\rho\vec{f}\cdot\vec{n})\mathrm{d}s$$

or

$$\int_V \frac{\partial\rho}{\partial t}\mathrm{d}V = -\int_S (\rho\vec{f}\cdot\vec{n})\mathrm{d}s$$

i.e., a positive flux $\implies$ a loss of "mass."

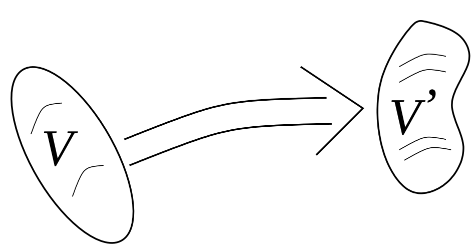
Now we apply the divergence theorem to convert the integral of the vector field $\rho\vec{f}$ on the surface S to a volume integral:

$$\int_V \frac{\partial\rho}{\partial t}\mathrm{d}V = -\int_V [\vec{\nabla}\cdot(\rho\vec{f})]\mathrm{d}V$$

Letting the volume V shrink, we have

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{f})$$

Now follow the motion of V to V' in time δt:



The boundary deforms, but it always contains the same points.

We wish to calculate $\mathrm{d}\rho/\mathrm{d}t$, which is the rate of change of ρ **as the volume moves**:

$$\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial\rho}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{\partial\rho}{\partial x_3}\frac{\mathrm{d}x_3}{\mathrm{d}t} \\
&= -\vec{\nabla}\cdot(\rho\vec{f}) \;+\; (\vec{\nabla}\rho)\cdot\vec{f} \\
&= -(\vec{\nabla}\rho)\cdot\vec{f} - \rho\vec{\nabla}\cdot f + (\vec{\nabla}\rho)\cdot\vec{f} \\
&= -\rho\vec{\nabla}\cdot\vec{f}
\end{aligned}$$

Note that the number of points in V is

$$N = \rho V$$

Since points are neither created nor destroyed we must have

$$\frac{\mathrm{d}N}{\mathrm{d}t} \;=\; V\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\frac{\mathrm{d}V}{\mathrm{d}t} \;=\; 0.$$

Thus, by our previous result,

$$-\rho V\vec{\nabla}\cdot\vec{f} = -\rho\frac{\mathrm{d}V}{\mathrm{d}t}$$

or

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \vec{\nabla}\cdot\vec{f}$$

This is called the *Lie derivative*.

We shall next arrive at the following main results by example:

- $\vec{\nabla}\cdot\vec{f} = 0 \Rightarrow$ volumes in phase space are conserved. Characteristic of conservative or Hamiltonian systems.
- $\vec{\nabla}\cdot\vec{f} < 0 \Rightarrow \mathrm{d}V/\mathrm{d}t < 0 \Rightarrow$ volumes in phase space contract. Characteristic of dissipative systems.

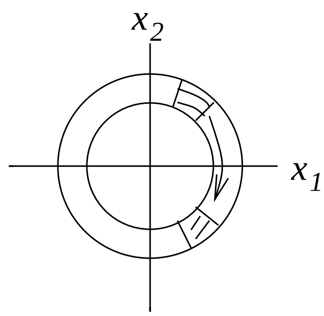
We use the example of the pendulum:

$$\begin{aligned}
\dot{x}_1 &= x_2 &&= f_1(x_1, x_2) \\
\dot{x}_2 &= -\frac{g}{l}\sin x_1 &&= f_2(x_1, x_2)
\end{aligned}$$

Calculate

$$\vec{\nabla} \cdot \vec{f} \;=\; \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} \;=\; 0 + 0$$

Pictorially



Note that the area is conserved.

Conservation of areas holds for *all* conserved systems. This is conventionally derived from Hamiltonian mechanics and the canonical form of equations of motion.

In conservative systems, the conservation of volumes in phase space is known as *Liouville's theorem.*

4 Damped oscillators and dissipative systems

4.1 General remarks

We have seen how conservative systems behave in phase space.
What about dissipative systems?

What is a fundamental difference between dissipative systems and conservative systems, aside from volume contraction and energy dissipation?

- Conservative systems are invariant under time reversal.
- Dissipative systems are not; they are *irreversible.*