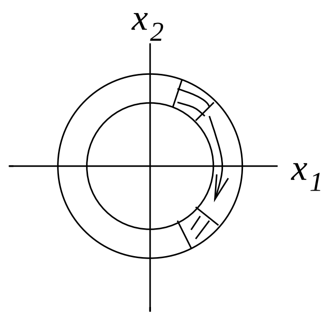
Calculate

$$\vec{\nabla} \cdot \vec{f} \ = \ \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} \ = \ 0 + 0$$

Pictorially



Note that the area is conserved.

Conservation of areas holds for *all* conserved systems. This is conventionally derived from Hamiltonian mechanics and the canonical form of equations of motion.

In conservative systems, the conservation of volumes in phase space is known as *Liouville's theorem.*

4 Damped oscillators and dissipative systems

4.1 General remarks

We have seen how conservative systems behave in phase space.
What about dissipative systems?

What is a fundamental difference between dissipative systems and conservative systems, aside from volume contraction and energy dissipation?

- Conservative systems are invariant under time reversal.
- Dissipative systems are not; they are *irreversible.*

Consider again the undamped pendulum:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2 \sin\theta = 0.$$

Let $t \to -t$ and thus $\partial/\partial t \to -\partial/\partial t$.
There is no change—the equation is *invariant* under the transformation.

The fact that most systems are dissipative is obvious if we run a movie backwards (ink drop, car crash, cigarette smoke...)

Formally, how may dissipation be represented? **Include terms proportional to odd time derivatives.**, i.e., break time-reversal invariance.

In the linear approximation, the damped pendulum equation is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega^2\theta = 0$$

where

$$\begin{aligned} \omega^2 &= g/l \\ \gamma &= \text{damping coefficient} \end{aligned}$$

The sign of γ is chosen so that positive damping is opposite the direction of motion.

How does the energy evolve over time? As before, we calculate

$$\begin{aligned} \text{kinetic energy} &= \frac{1}{2}ml^2\dot{\theta}^2 \\ \text{potential energy} &= mlg(1-\cos\theta) \simeq mlg\left(\frac{\theta^2}{2}\right) \end{aligned}$$

where we have assumed $\theta \ll 1$ in the approximation.

Summing the kinetic and potential energies, we have

$$\begin{aligned} E(\theta, \dot{\theta}) &= \frac{1}{2}ml^2\left(\dot{\theta}^2 + \frac{g}{l}\theta^2\right) \\ &= \frac{1}{2}ml^2(\dot{\theta}^2 + \omega^2\theta^2) \end{aligned}$$

Taking the time derivative,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}ml^2(2\dot{\theta}\ddot{\theta} + 2\omega^2\dot{\theta}\theta)$$

Substituting the damped pendulum equation for $\ddot{\theta}$,

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= ml^2[\dot{\theta}(-\gamma\dot{\theta} - \omega^2\theta) + \omega^2\dot{\theta}\theta] \\ &= -ml^2\gamma\dot{\theta}^2 \end{aligned}$$

Take $ml^2 = 1$. Then

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\gamma\dot{\theta}^2$$

Conclusion:

$\gamma = 0 \Rightarrow$ Energy conserved (no friction)
$\gamma > 0 \Rightarrow$ friction (energy is dissipated)
$\gamma < 0 \Rightarrow$ energy increases without bound

4.2 Phase portrait of damped pendulum

Let $x = \theta$, $y = \dot{\theta}$.
Then

$$\begin{aligned} \dot{x} &= \dot{\theta} = y \\ \dot{y} &= \ddot{\theta} = -\gamma\dot{\theta} - \omega^2\theta = -\gamma y - \omega^2 x \end{aligned}$$

or

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues of the system are solutions of

$$(-\lambda)(-\gamma - \lambda) + \omega^2 = 0$$

Thus

$$\lambda = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 4\omega^2}$$

Assume $\gamma^2 \ll \omega^2$ (i.e., weak damping, small enough to allow oscillations). Then the square root is complex, and we may approximate λ as

$$\lambda = -\frac{\gamma}{2} \pm i\omega$$

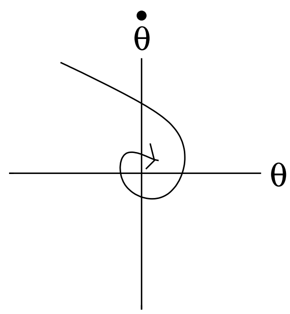
The solutions are therefore exponentially damped oscillations of frequency ω:

$$\theta(t) = \theta_0 e^{-\gamma t/2} \cos(\omega t + \phi)$$

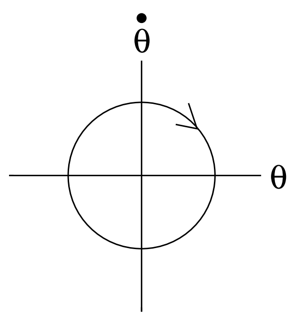
θ_0 and ϕ derive from the initial conditions.

There are three generic cases:

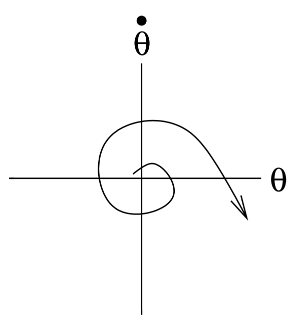
- for $\gamma > 0$, trajectories spiral inwards and are **stable**.



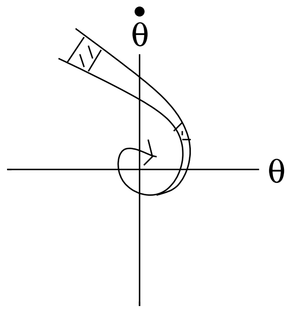
- for $\gamma = 0$, trajectories are **marginally stable** periodic oscillations.



- for $\gamma > 0$, trajectories spiral outwards and are **unstable**.



It is obvious from the phase portraits that the damped pendulum contracts areas in phase space:



We quantify it using the Lie derivative,

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \vec{\nabla} \cdot \vec{f}$$

which yields

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} \ = \ 0 - \gamma \ = \ -\gamma \ < \ 0$$

The inequality not only establishes area contraction, but γ gives the rate.

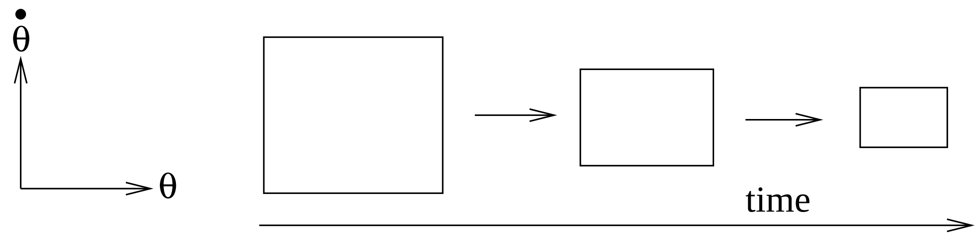
4.3 Summary

Finally, we summarize the characteristics of dissipative systems:

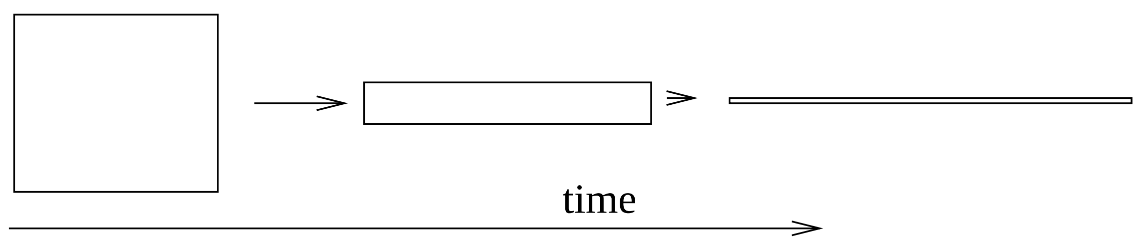
- Energy not conserved.
- Irreversible.
- Contraction of areas (volumes) in phase space.

Note that the contraction of areas is not necessarily simple.

In a 2-D phase space one might expect



However, we can also have



i.e., we can have expansion in one dimension and (a greater) contraction in the other.

In 3-D the stretching and thinning can be even stranger!