Bifurcations: baby normal forms.

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Abstract

The normal forms for the various bifurcations that can occur in a one dimensional dynamical system ($\dot{x} = f(x, r)$) are derived via local approximations to the governing equation, valid near the critical values where the bifurcation occurs. The derivations are non-rigorous.

Contents

1 Introduction.

Consider the simple one-dimensional dynamical system

$$\frac{dx}{dt} = f(x, r), \tag{1.1}$$

where we will assume that $f = f(x, r)$ is a smooth function, and r is a parameter. We wish to study the possible bifurcations for this system, as the parameter r varies. Because the phase portrait for a 1-D system is fully determined by its critical (equilibrium) points, we need only study what happens to the critical points. Bifurcations will (only) occur as these points are created, destroyed, collide, or change stability. For higher dimensional systems, the critical points alone do not determine the phase portrait. However, the bifurcations we study here can still occur, and are important. Furthermore, the normal forms we develop here still apply. Thus:

Consider some critical point $x = x_0$ (occurring for a value $r = r_0$) i.e. $\boxed{f(x_0, r_0) = 0.}$ Then ask: $\boxed{\textbf{When is } (x_0, r_0) = 0 \textbf{ a bifurcation point?}}$ A **necessary condition** is:

$$f_x(x_0, r_0) = 0. \tag{1.2}$$

Why? Because otherwise the implicit function theorem would tells us that: In a neighborhood of (x_0, r_0), the critical point equation $f(x, r) = 0$ has a unique (smooth) solution $x = X(r)$, which satisfies $X(r_0) = x_0$. Thus no critical points would be created, destroyed or collide at (x_0, r_0). Further, obviously: no change of stability can occur if $f_x(x_0, r_0) \neq 0$.

$\boxed{\text{Without loss of generality, in what follows we will assume that } x_0 = r_0 = 0.}$

Remark 1 *From equation (1.2) we see that bifurcations and undecided (linearized) stability are intimately linked. This is true not just for 1-D systems, and (in fact) applies even for bifurcations that do not involve critical points.*

Remark 2 *For higher dimensional systems (where x and f are vectors of the same dimension), f_x is a square matrix, and the condition (1.2) gets replaced by f_x* is singular. *The proof of this is essentially the same as above, via the implicit function theorem (even in infinite dimensions, as long as an appropriate version of the implicit function theorem applies,[1] the result is true). We*

[1] In infinite dimensions the implicit function theorem may not apply.

point out that, as long as f_x has a one-dimensional kernel (zero is a multiplicity one eigenvalue), most of what follows next applies for higher dimensional systems as well.

2 Saddle Node bifurcations.

Given a critical point, say $(x, r) = (0, 0)$, with $f(0, 0) = f_x(0, 0) = 0$, the most generic situation is that where $f_r(0, 0) \neq 0$ and $f_{xx}(0, 0) \neq 0$. By appropriately re-scaling x and r in (1.1) — if needed, we can thus assume that:

$$f(0,0) = f_x(0,0) = 0, \quad f_r(0,0) = 1, \quad \text{and} \quad f_{xx}(0,0) = -2. \tag{2.3}$$

Remark 3 *For arbitrary dynamical systems (such as (1.1)), we have to be careful about assuming that anything is exactly zero.* Situations where something vanishes exactly are (generally) structurally unstable*, since arbitrarily small perturbations will destroy them. To (safely) make such assumptions, we need extra information about the system: information that restricts the possible perturbations — in such a way that whatever vanishes, remains zero when the system is perturbed.*

Remark 4 *In view of the prior remark, the reader may very well wonder:* How do we justify the assumptions above, namely: $f(0, 0) = f_x(0, 0) = 0$? *The answer is that:* It is the full set of assumptions in (2.3) that is structurally stable, not just the first two. *We prove this next. Since (2.3) characterizes the saddle node bifurcations (we show this later), this will prove that:* Saddle Node bifurcations are structurally stable.

Proof: *First, to show that the first two assumptions (when alone) are structurally unstable, consider the example: $f = r^2 + x^2$, with the critical point $(0, 0)$. Then change f to $f = r^2 + x^2 + 10^{-30}$, which causes the critical point to cease to exist. This example illustrates the fact that:*

Isolated critical points[2] are structurally unstable, thus not (generally) interesting.

Second: imagine now that f depends on some extra parameter $f = f(x, r, h)$, such that the assumptions in (1.2) apply for $(x, r, h) = (0, 0, 0)$ — here h small and nonzero produces an "arbitrary" (smooth) perturbation to the dynamical system in (1.1). Consider now the system of equations:

$$f(x, r, h) = 0, \quad \textit{and} \quad f_x(x, r, h) = 0. \tag{2.4}$$

[2]These are points such that there is a neighborhood in (x, r) space where there is no other critical point.

Now $(0,0,0)$ *is a solution to this system, and the Jacobian matrix*

$$J = \begin{pmatrix} f_x(0,0,0) & f_r(0,0,0) \\ f_{xx}(0,0,0) & f_{xr}(0,0,0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & f_{xr}(0,0,0) \end{pmatrix} \tag{2.5}$$

is non-singular there. Thus the implicit function theorem guarantees that there is a (unique) smooth curve of solutions $x = x(h)$ *and* $r = r(h)$ *to (2.4), with* $x(0) = 0$ *and* $r(0) = 0$*. Along this curve, for* h *small enough, it is clear that:* $f_r(x,r,h) \neq 0$ *and* $f_{xx}(x,r,h) \neq 0$*. Thus, modulo normalization, (2.3) is valid along the curve — for* h *small enough. This finishes the proof.*

Remark 5 *In the proof of structural stability in the prior remark, we assumed that the perturbations to the dynamical system in (1.1) had the form*

$$\frac{dx}{dt} = f(x,r,h), \tag{2.6}$$

with the dependence in the "extra" parameter h *being smooth. This sounds reasonable, but (clearly) it does not cover all possible (imaginable or non-imaginable) perturbations. For example, we could consider "perturbations" of the form*

$$\frac{dx}{dt} = f(x,r) + h\frac{d^2x}{dt^2}. \tag{2.7}$$

What "small" means in this case is not easy to state (and we will not even try here). However, this example should make it clear that: when talking about structural stability, for the concept to even make sense, the dynamical system must be thought as belonging to some "class" — within which the idea of "close" makes sense. *Further, the answer to the question: is this system structurally stable? will be a function of the class considered.*

Let us now get back to the system in (1.1), with the assumptions in (2.3), and let us study the bifurcation that occurs in this case: the Saddle Node bifurcation.

We proceed formally first, by expanding in Taylor series and writing the equation in the form

$$\frac{dx}{dt} = r - x^2 + O(r^2,\, rx, x^3), \tag{2.8}$$

where all the information in (2.3) has been used. We now look at this equation in a small (rectangular) neighborhood of the origin, characterized by

$$|x| < \epsilon \quad \text{and} \quad |r| < \epsilon^2, \tag{2.9}$$

where $0 < \epsilon \ll 1$. Then the first two terms on the right in (2.8) are $O(\epsilon^2)$, while the rest is $O(\epsilon^3)$. We thus argue that the behavior of the system in the neighborhood given by (2.9) is well approximated by the equation

$$\frac{dx}{dt} = r - x^2. \tag{2.10}$$

This is the $\boxed{\textbf{Normal form for a Saddle Node bifurcation}}$ — see Strogatz book for a description of its behavior.

Remark 6 *A natural question here is:* **Why the scaling in (2.9)?** *Such a question can only be answered "after the fact", with the answer being (basically) "because it works". Namely, after we have figured out what is going on, we can explain why the scaling in (2.9) is the right one to do. As follows: at a Saddle Node bifurcation — say, at $(x, r) = (0, 0)$ — a branch of critical point solutions — say $x = X_1(r)$ — turns "back" on itself.*[3] *Thus, on one side of the value $r = 0$, no critical point exist, while on the other side two are found, say at: $x = X_1(r)$ and $x = X_2(r)$. Locally, these two curves can be joined into a single one by writing $r = R(x)$. Then $r = R(x)$ has either a maximum (or a minimum) at $x = 0$. Hence it can, locally, be approximated by a parabola. Hence the scaling in (2.9) is the right one. Any other scaling would miss the fact that we have a branch of critical points turning around.*

Those with a mathematical mind will probably not be very satisfied with this explanation. For them, the theorem below might do the trick. However, note that this theorem is just a proof that equation (2.10) is the right answer, showing that (2.9) works. It does not give any reason (or method) that would justify (2.9) "a priori". Simply put: advance in science and mathematics requires places at which "insight" is needed, and (2.9) is an example of this; perhaps a very simple example, but one nonetheless.

Theorem 1 *With the hypothesis in equation (2.9), there exists a neighborhood of the origin, and there a smooth coordinate transformation $(x, t) \to (X, T)$ of the form*

$$X = x\,\Phi(x) \quad \text{and} \quad \frac{dT}{dt} = \Psi(x, r), \tag{2.11}$$

such that (1.1) is transformed into $\dfrac{dX}{dT} = r - X^2$ — that is, the normal form in equation (2.10). Furthermore: $\Phi(0) = 1$ and $\Psi(0, 0) = 1$ — thus: $X \approx x$ and $T \approx t$ close to the origin.

[3] Note that this is the reason that this type of bifurcation is also known by the name of **turning point bifurcation.**

IMPORTANT: *the definition for the transformed time is meant to be done along the solutions. That is: in the equation* $\frac{dT}{dt} = \Psi(x, r)$, $x = x(t)$ *is a solution of equation (1.1).*

Proof: Using the implicit function theorem, we see that $f(x, r) = 0$ has a unique (also smooth) solution $r = R(x)$ in a neighborhood of the origin: $\boxed{f(x, R(x)) \equiv 0,}$ which satisfies $R(0) = 0$. It is easy to see that $(dR/dx)(0) = 0$ and $(d^2R/dx^2)(0) = 2$ also apply. Thus $\boxed{R = x^2\,\Phi(x)^2,}$ **where** Φ **is smooth and** $\Phi(0) = 1$ — this is the Φ which appears in equation (2.11).

Because $f(x, R(x)) \equiv 0$, we can write $\boxed{f(x, r) = \alpha(x, r)\,(r - R(x)) = \alpha(x, r)\,(r - X^2),}$ where α is smooth and does not vanish near the origin — in fact: $\boxed{\alpha(0, 0) = 1.}$

Define now $\boxed{\Psi = (\Phi + x\Phi')\,\alpha,}$ where the prime indicates differentiation with respect to x. It is then easy to check that $\Psi(0, 0) = 1$, and that with this definition (2.11) yields $\frac{dX}{dT} = r - X^2$.

QED.

Problem 1 *Implement a reduction to normal form, along lines similar to those used in theorem* **1**, *by formally expanding the coordinate transformation up to* $O(\epsilon^2)$ *— where* ϵ *is as in equation (2.9). To do so, write the dynamical system in the expanded form*

$$\frac{dx}{dt} = \underbrace{r - x^2}_{O(\epsilon^2)} + \underbrace{a\,r\,x + b\,x^3}_{O(\epsilon^3)} + O(\epsilon^4), \tag{2.12}$$

where a *and* b *are constants. Then expand the transformation*

$$x = X + \alpha\,X^2 + O(\epsilon^3), \tag{2.13}$$

$$\frac{dt}{dT} = 1 + \beta\,X + O(\epsilon^2), \tag{2.14}$$

and find what values the coefficients α *and* β *must take so that*

$$\frac{dX}{dT} = r - X^2 + O(\epsilon^4). \tag{2.15}$$

This process can be continued so as to make the error term in (2.15) as high an order in ϵ *as desired — provided that* f *in (1.1) has enough derivatives. We point out here that:* **theorem 1 requires** f **to have only second order continuous derivatives to apply.** *By contrast, the process here requires progressively higher derivatives to exist — it, however, has the advantage of giving explicit formulas for the transformation.*

3 Transcritical bifurcations.

We now go back to the considerations in the introduction (section 1), and add one extra hypothesis at the bifurcation point $(x_0, r_0) = (0, 0)$. Namely: we **assume that there is a smooth branch $x = \chi(r)$ of critical points that goes through the bifurcation point.** Taking successive derivatives of the identity $f(\chi(r), r) \equiv 0$, and evaluating them at $r = 0$ (where $\chi = f = f_x = 0$), we obtain:

$$f_r(0,0) = 0 \quad \text{and} \quad f_{rr}(0,0) = -\mu^2 f_{xx}(0,0) - 2\,\mu\, f_{xr}(0,0), \tag{3.16}$$

where $\mu = \dfrac{d\chi}{dr}(0)$. As before, we assume that the coefficients for which we have no information are non-zero, and normalize (by scaling r and x in equation (1.1), if needed) so that: $f_{xx}(0,0) = -2$ and $f_{rx}(0,0) = 1$. Thus, **at** $(x,r) = (0,0)$**, we have:**

$$f = f_x = f_r = 0, \quad f_{xx} = -2, \quad f_{xr} = 1, \quad \text{and} \quad f_{rr} = 2\,a, \tag{3.17}$$

where a is a constant. In fact, (3.16) and (3.17), show that $a = \mu^2 - \mu = (\mu - 1/2)^2 - 1/4$. Thus $1 + 4\,a = (2\,\mu - 1)^2 \geq 0$. We do not know what the exact value of μ is, however, as usual (for generality) we exclude the equal sign in this last inequality as "too special". Thus:

$$\textbf{Assume } \boxed{1 + 4\,a > 0.} \tag{3.18}$$

As in the case of the Saddle Node bifurcation, the next step is to use (3.17) to expand the equation (1.1). This yields:

$$\frac{dx}{dt} = r\,x - x^2 + a\,r^2 + O(x^3,\, r\,x^2,\, r^2\,x, r^3). \tag{3.19}$$

We now assume[4] that both r and x are small, of size $O(\epsilon)$, where $0 < \epsilon \ll 1$. Then, keeping up to terms of $O(\epsilon^2)$ on the right (leading order) in (3.19), we obtain the equation:

$$\frac{dx}{dt} = r\,x - x^2 + a\,r^2 = -(x - \sigma_1\,r)\,(x - \sigma_2\,r), \tag{3.20}$$

where $\sigma_1 = \dfrac{1}{2}\left(1 + \sqrt{1 + 4\,a}\right)$ and $\sigma_2 = \dfrac{1}{2}\left(1 - \sqrt{1 + 4\,a}\right)$. In terms of the variables $X = x - \sigma_2\,r$ and $R = \sqrt{1 + 4\,a}\,r$, this last equation takes the form:

$$\frac{dX}{dt} = R\,X - X^2, \tag{3.21}$$

which is the $\boxed{\textbf{Normal form for a Transcritical bifurcation.}}$

[4] Compare this with (2.9).

Remark 7 *The hypothesis $1+4\,a>0$ in (3.18) is very important. For, write equation (3.19) in the form:*

$$\frac{dx}{dt} = -\left(x - \frac{1}{2}\,r\right)^2 + \frac{1+4\,a}{4}\,r^2 + O(x^3,\, r\,x^2,\, r^2\,x, r^3). \tag{3.22}$$

*Then, if $1+4\,a<0$, the leading order terms on the right in this equation would be a negative definite quadratic form. This would imply that $(x, r) = (0, 0)$ is the only critical point in a neighborhood of the origin — i.e. that $(x, r) = (0, 0)$ is an isolated critical point. As explained in remark **4**, such points are (generally) of little interest.*

On the other hand, $1+4\,a=0$ would lead to a double root of the right hand side in (3.19) (at leading order). In principle this can be interpreted as a "limit case" of the transcritical bifurcation, where the two branches of critical points that cross at the origin, become tangent there. However: this is an extremely structurally unstable situation, where the local details of what actually happens are controlled by high order terms — hence, again, this is a situation of little (general) interest.

Theorem 2 *If the function $f = f(x, r)$ is sufficiently smooth, the assumptions in equations (3.17) and (3.18) guarantee that the $f = f(x, r) = 0$ has (exactly) two branches of solutions in a neighborhood of the origin. Furthermore, let this branches be given by $x = \chi_1(r)$ and $x = \chi_2(r)$. Then $\chi_1(r) = \sigma_1\,r + O(r^2)$ and $\chi_2(r) = \sigma_2\,r + O(r^2)$.*

Sketch of the proof: The calculations leading to equations (3.19) and (3.20) show that:

$$f(x, r) = -(x-\sigma_1\,r)\,(x-\sigma_2\,r) + O(x^3,\, r\,x^2,\, r^2\,x,\, r^3). \tag{3.23}$$

Let $x = r\,X$. Then

$$f(x, r) = -r^2\,(X-\sigma_1)\,(X-\sigma_2) + r^3\,O(X^3,\, X^2,\, X). \tag{3.24}$$

Thus, $\boxed{g = g(X, r) = -\dfrac{f}{r^2}}$ satisfies:

$$g(X, r) = (X-\sigma_1)\,(X-\sigma_2) + r\,h(X, r), \tag{3.25}$$

where h is some non-singular function. We note now that:

$$g(\sigma_p, 0) = 0 \quad \text{and} \quad g_X(\sigma_p, 0) = (\sigma_p - \sigma_q) \neq 0, \tag{3.26}$$

where $\{p, q\} = \{1, 2\}$. Then the implicit function theorem guarantees that there exist smooth solutions $X = X_n(r)$ to the equations: $\boxed{g(X_n, r) = 0 \text{ and } X_n(0) = \sigma_n}$ — where $n = 1$ or $n = 2$. Then $\chi_n = r\, X_n$ — for $n = 1, 2$ — are the two functions in the theorem statement.

Why are there no other solutions? Well, once we have χ_1 and χ_2, we can write $f = (x - \chi_1)(x - \chi_2)\psi$, where $\psi = \psi(x, r)$ does not vanish at the origin — in fact: $\psi(0, 0) = f_{xx}(0, 0) = -2$.

QED.

The arguments made to obtain equations (3.17) and (3.18) depend on the existence of the smooth branch of critical points $x = \chi(r)$. But the existence of this branch is not then used in the arguments leading to the normal form (3.21). We explicitly exploit this existence in what follows below, and use it to get a better handle on transcritical bifurcations. Thus, without loss of generality:[5]

$$\boxed{\textbf{Assume that } \chi \equiv 0.} \tag{3.27}$$

Then we can write $\boxed{f = x\, G(x, r),}$ where $\boxed{G(0, 0) = 0}$ — since $f_x(0, 0) = 0$. Other than this, we assume that G is "generic", so that its derivatives do not vanish. In particular, we normalize the first order derivatives so that $\boxed{G_r(0, 0) = -G_x(0, 0) = 1}$ — this normalization is consistent with the one used in (3.17), where we must take $a = 0$.

At this point we can invoke the implicit function theorem, that tells us that there is a function $x = z(r)$ such that $G(z, r) = 0$, with $z(0) = 0$ and $\boxed{dz/dr(0) = 1}$ — note that, in this case $\sigma_1 = 1$ and $\sigma_2 = 0$. Again, we use this function to factor G in the form $\boxed{G = (x - z(r))\, H(x, r),}$ where $\boxed{H(0, 0) = -1.}$ It follows then that we can write equation (1.1) in the form:

$$\frac{1}{H}\frac{dx}{dt} = -z(r)\, x + x^2. \tag{3.28}$$

Thus, if we introduce a new time T by $\boxed{dT/dt = -H,}$ and change parameter[6] $\boxed{r \to R = z(r),}$ the equation is transformed into its **Normal Form:**

$$\frac{dx}{dT} = R\, x - x^2. \tag{3.29}$$

The above is, clearly, the **equivalent of theorem 1 for transcritical bifurcations: a proof of the existence of a local transformation into normal form.**

[5]If needed, the change of variables $x \to x - \chi$ will do the trick.

[6]Note that, for r and x small, $R \approx r$ and $T \approx t$. Thus both R and T are acceptable new variables.

Remark 8 *Note that, because G above is "generic", the* **situation is structurally stable.** *However, this depends on the assumption that there is a branch of solutions. Transcritical bifurcations are not structurally stable without an assumption of this type.*

Problem 2 *Assume that equation (3.27), and the normalizations immediately below it, apply. Then, for x and r both small and $O(\epsilon)$ — where $0 < \epsilon \ll 1$ — implement a reduction to normal form, by formally expanding the coordinate transformation up to two orders in ϵ. To do so, write the dynamical system in the expanded form*

$$\frac{dx}{dt} = \underbrace{r\,x - x^2}_{O(\epsilon^2)} + \underbrace{b_0\,x^3 + b_1\,r\,x^2 + b_2\,r^2\,x}_{O(\epsilon^3)} + O(\epsilon^4), \tag{3.30}$$

where b_0, b_1, and b_2 are constants. Then expand the transformation

$$\frac{dt}{dT} = 1 + \beta_0\,x + \beta_1\,R + O(\epsilon^2), \tag{3.31}$$

$$r = R + \gamma\,R^2 + O(\epsilon^3), \tag{3.32}$$

and find what values the coefficients β_0, β_1, and γ must take so that

$$\frac{dx}{dT} = R\,x - x^2 + O(\epsilon^4). \tag{3.33}$$

This process can be continued so as to make the error term in (3.33) of arbitrarily high order in ϵ — provided that f in (1.1) has enough derivatives. We point out here that: **the derivation leading to equation (3.29) requires f to have only second order continuous derivatives to apply.** *By contrast, the process here requires progressively higher derivatives to exist — it, however, has the advantage of giving explicit formulas for the transformation.*

Problem 3 *In problem 3.2.6 in the book by Strogatz, a process somewhat analogous to the one in problem* **2** *is introduced. Basically, Strogatz tells you to do the following:*
Consider the system

$$\frac{dx}{dt} = R\,x - x^2 + a\,x^3 + O(x^4), \tag{3.34}$$

where $R \neq 0$ and a are constants. Introduce now a transformation (expanded) of the form

$$x = X + b\,X^3 + O(X^4), \tag{3.35}$$

where b is a constant. Then show that b can be selected so that the equation for X has the form

$$\frac{dX}{dt} = R\,X - X^2 + O(X^4). \tag{3.36}$$

Thus the third order power is removed. The process can be generalized to remove arbitrarily high powers of X from the equation.

Question: This process is simpler than the one employed in problem **2**: it involves neither transforming the independent variable t, nor the parameter R. **Why is it not appropriate for reducing an equation to normal form near a transcritical bifurcation?**

4 Pitchfork bifurcations.

We now go back to the considerations in the introduction (section 1), and add two extra hypotheses at the bifurcation point $(x_0,\, r_0) = (0,\, 0)$, one of them being the same one that was introduced in section 3 for the transcritical bifurcations. Namely, we **assume that:**

A. There is a smooth branch $x = \chi(r)$ of critical points that goes through the bifurcation point $(0,\, 0)$.

B. The problem has right-left symmetry across the branch of critical points $x = \chi(r)$. Specifically, there is smooth bijection $x \to X = X(x,\, r)$, valid in a neighborhood of the branch $x = \chi$, such that:
— Equation (1.1) is invariant under the transformation: $\dot{X} = f(X,\, r)$.
— χ is a fixed curve for the transformation: $X(\chi(r),\, r) = \chi(r)$.
— $x < \chi(r) \implies X > \chi(r)$ and $x > \chi(r) \implies X < \chi(r)$.

Without any real loss of generality, assume that $f(x,\, r)$ is an odd function of x. Then $\chi = 0$, $X = -x$, and (1.1) becomes:

$$\frac{dx}{dt} = x\,g(\zeta,\, r), \quad \text{where} \quad \zeta = x^2. \tag{4.37}$$

The bifurcation condition (1.2) yields $g(0,\,0) = 0$. Other than this, we assume that g is generic. After appropriate re-scaling of the variables, we thus have

$$g(0,\,0) = 0, \quad g_r(0,\,0) = 1, \quad \text{and} \quad g_\zeta(0,\,0) = \nu = \pm 1. \tag{4.38}$$

Note that the sign of $g_\zeta(0,\,0)$ cannot be changed by scalings!

Problem 4 *Expand g in (4.37) in powers of ζ and r. Show that, in a small neighborhood of the origin (of appropriate shape — see (2.9) and (3.19 – 3.20)), the leading order terms in the equation reduce to the* **normal form for a pitchfork bifurcation:** $\boxed{\dfrac{dx}{dt} = r\,x + \nu\,x^3.}$

Problem 5 *In a manner analogous to the ones in problems* **1** *(saddle-node bifurcations) and* **2** *(transcritical bifurcations) introduce new variables (via formal expansions) $x \to X$, $t \to T$, and $r \to R$, that reduce equation (4.37) to* **normal form:** $\boxed{\dfrac{dX}{dT} = R\,X + \nu\,X^3.}$

HINT: First • Expand g in a Taylor series $\boxed{g = r + \nu\,\zeta + a_2\,r^2 + a_1\,r\,\zeta + a_0\,\zeta^2 + \dots}$ and substitute this expansion into the equation. **Second** • Assume an appropriate size scaling for the variables x and r in terms of a small parameter $0 < \epsilon \ll 1$. This scaling should be consistent with the normal form for the equation.[7] It is very important since it assures that the ordering in the expansions is kept straight, without higher order terms being mixed with lower order ones. **Third** • Introduce expansions for $R = R(r) = r + o(r)$ and $dT/dt = H(x^2,\,r) = 1 + o(1)$. **IMPORTANT:** Notice that, because of the symmetry in the equation, it must be that $x = X$ and the expansion for dT/dt must involve even powers of x only. **Fourth** • Substitute the expansions in the equation, and select the coefficients to eliminate the higher orders beyond the normal form. **Carry this computation to ONE ORDER ONLY: What are the dominant terms in the expansions, beyond $R \sim r$ and $dT/dt \sim 1$.**

Problem 6 *Prove that a transformation with the properties stated in problem* **5** *actually exists — this in a way similar to the one used in theorem* **1** *for saddle-node bifurcations, and above equation (3.29) for transcritical bifurcations.*

HINT: Show that $g(\zeta,\,r) = 0$ has a solution of the form $\zeta = -\nu\,R(r)$. Use this solution to "factor" g as a product, and substitute the result into the equation. It should then be obvious how to proceed.

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[7] It is the same scaling required by problem **4**.

5 Problem Answers.

The problem answers will be handed out with the answers to the problem sets.