2.29 Numerical Marine Hydrodynamics
Spring 2007

Course Staff:

Instructor: Prof. Henrik Schmidt

OCW Web Site: http://ocw.mit.edu/OcwWeb/Mechanical-Engineering/2-29Spring-2003/CourseHome/index.htm

Units: (3-0-9)

Lectures: Tuesday/Thursday 11:00 a.m.— 12:30 p.m.

Instructor: Prof. Henrik Schmidt, Tuesday 4-5 pm.

Summary:
Introductory MATLAB. Introduction to numerical methods: number representation and errors, interpolation, differentiation, integration, systems of linear equations, Fourier interpolation and transforms. Differential equations: partial and ordinary differential equations, elliptic and parabolic differential equations. Solution of differential equations by numerical integration, finite difference methods, finite element methods, boundary element methods and panel methods.

Class Schedule

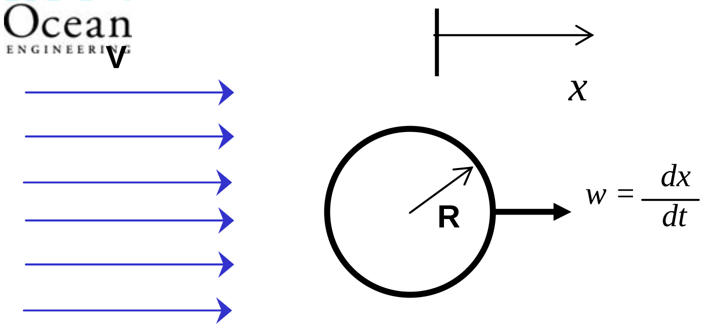
Tuesday 11 - 12:30	Thursday 11 - 12:30	Lecture	Reading Chapra & Canale
6-Feb		Introduction to Numerical Methods in Engineering	PT 1.1-1.3, 1.1 - 1.2
	8-Feb	Number representations. Errors of numerical operations. Recursion	3.1 - 3.4
13-Feb		Error analysis. Error propagation. Condition numbers	4.1 - 4.5
	15-Feb	Roots of non-linear equations. General/Bisection/Secant/Newton-Raphson methods	5.1 -5.4,6.1 - 6.5
	22-Feb	Linear systems. Gaussian elimination	9.1 - 9.8
27-Feb		Linear systems. Multiple right-hand-sides. LU facttorization	10.1 -10.3
	1-Mar	Special matrices. Examples	11.1
6-Mar		Linear systems. Iterative techniques. Gauss-Seidel.	11.2
	8-Mar	Root finding and Linear systems. Examples and Apllications	8.1-8.4, 12.1-12.4
13-Mar		Optimization. Curve fitting	13, 14, 17
	15-Mar	Interpolation. Polynomial interpolation. Lagrange polyn. Splines..	18.1 -18.6
20-Mar		Fourier interpolation. Fourier transforms.	19.1 -19.8
	22-Mar	Quiz 1	
3-Apr		Numerical Integration. Newton-Cotes. Gaussian quadratutre	21.1-21.2, 22.1-22.3
	5-Apr	Numerical Differentiation. Finite differences	23.1 - 23.5
10-Apr		Ordinary differential equations. Initial value problems. Euler's method.	25.1 - 25.2
	12-Apr	ODE-IVP. Runge-Kutta methods	25.3 - 23.5
	19-Apr	Higher order ODEs	Notes
24-Apr		ODE, Boundary value problems.	27.1 - 27.3
	26-Apr	Partial Differential equations. Introduction. Examples	P.T. 8.1 -8.2
1-May		PDEs. Elliptic Equations.	29.1 - 29.5
	3-May	PDEs. Parabolic Equations.	30.1 - 30.5
8-May		Finite Element methods	31.1 - 31.4
	10-May	Boundary Element methods. Panel methods (1)	Notes
15-May		Boundary Element methods. Panel methods (2)	Notes
	17-May	Quiz 2	

Numerical Marine Hydrodynamics

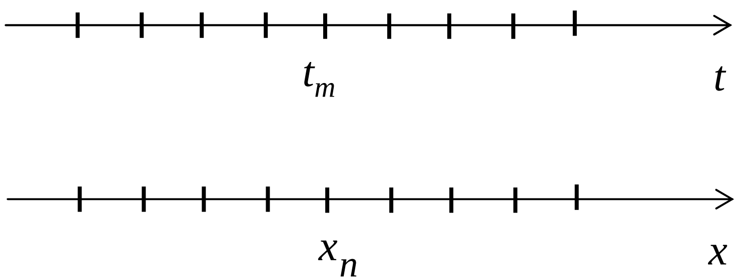
- Introduction to Numerical Hydrodynamics
  - Continuum and Discrete Representation
  - Particle Image Velocimetry Analysis
  - Navier-Stokes Equations
  - Differential equations
  - Integral equations
- Fundamentals of Digital Computing
  - Digital Computer Models
  - Convergence, accuracy and stability
  - Number representation
  - Arithmetic operations
  - Recursion algorithms
- Error Analysis
  - Error propagation – numerical stability
  - Error estimation
  - Error cancellation
  - Condition numbers

Digital Computer Models

Continuum Model



Discrete Model



$$t_m = t_0 + m\Delta t\ , \quad m = 0, 1, \ldots M-1$$

$$x_n = x_0 + n\,\Delta x\ , \quad n = 0, 1, \ldots N-1$$

$$\frac{dw}{dx} \simeq \frac{\Delta w}{\Delta x}\ , \quad \frac{dw}{dt} \simeq \frac{\Delta w}{\Delta t}$$

Differential Equation

$$L(p, w, x, t) = 0$$

Differentiation
Integration

Difference Equation

$$L_{mn}(p_{mn}, w_{mn}, x_n, t_m) = 0$$

System of Equations

$$\sum_{j=0}^{N-1} F_i(w_j) = B_i$$

Linear System of Equations

$$\sum_{j=0}^{N-1} A_{ij} w_j = B_i$$

Solving linear equations

Eigenvalue Problems

$$\overline{\overline{\mathbf{A}}}\mathbf{u} = \lambda\mathbf{u} \Leftrightarrow (\overline{\overline{\mathbf{A}}} - \lambda\overline{\overline{\mathbf{I}}})\mathbf{u} = \mathbf{0}$$

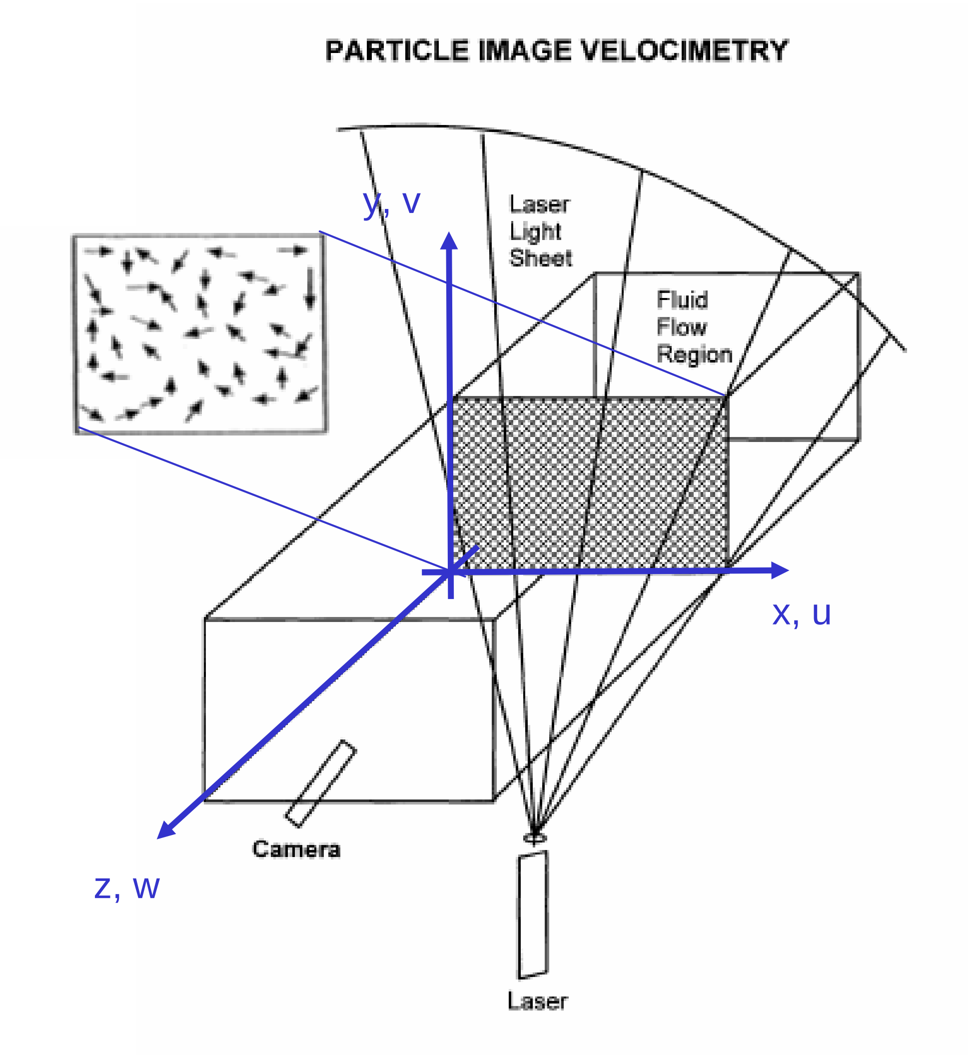
Non-trivial Solutions

$$\det(\overline{\overline{\mathbf{A}}} - \lambda\overline{\overline{\mathbf{I}}}) = 0$$

Root finding

$\boxed{\text{Accuracy and Stability => Convergence}}$

Particle Image Velocimetry



Conservation of Mass

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

$$w = \int_{z_0}^{z} \frac{\partial w}{\partial z} dz + w_0$$

Plane Flow Field

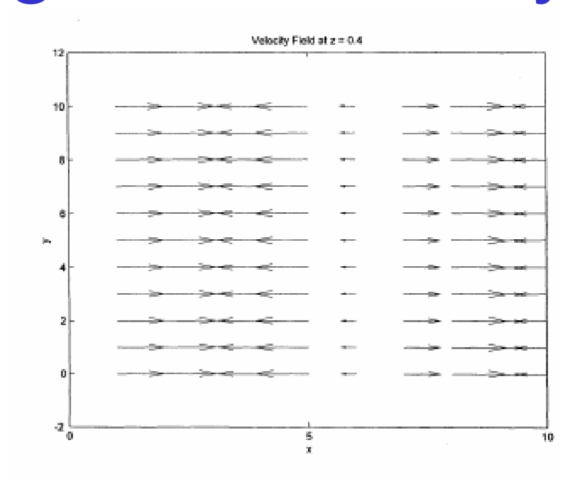
$$\begin{aligned} v &= 0 \\ \frac{\partial u}{\partial x} &= (3e^z - ze^z - 3)\cos x \\ &\Rightarrow \\ \frac{\partial w}{\partial z} &= -(3e^z - ze^z - 3)\cos x \end{aligned}$$

$$\begin{aligned} w = \int_0^z \frac{\partial w}{\partial z} dz &= -(3e^z - 3 - ze^z + e^z - 1 - 3z)\cos x \\ &= -(4e^z - 4 - ze^z - 3z]\cos x \end{aligned}$$

Particle Image Velocimetry

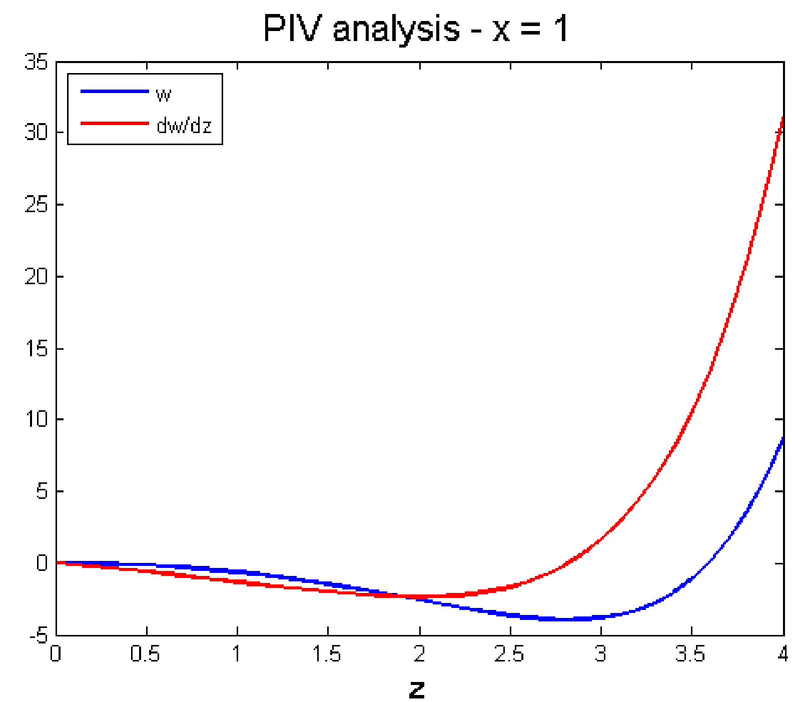
$$\frac{\partial w}{\partial z} = -(3e^z - ze^z - 3)\cos x$$

$$w = \int_0^z \frac{\partial w}{\partial z} dz = -(3e^z - 3 - ze^z + e^z - 1 - 3z)\cos x$$
$$= -(4e^z - 4 - ze^z - 3z]\cos x$$

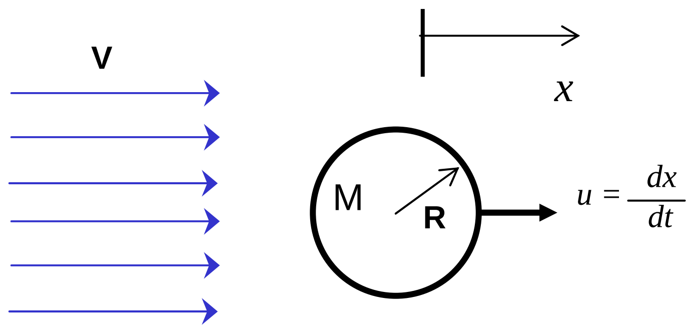


pvi.m

```
x=1;
z=[0:0.1:4];
dwdz=-(3*exp(z)-z .* exp(z) -3) * cos(x);
w=-(4* exp(z) -4 -z.*exp(z) -3*z) * cos(x);

figure
hold off
a=plot(z,w,'b');
set(a,'Linewidth',2);
hold on;
a=plot(z,dwdz,'r');
set(a,'Linewidth',2);
a=title(['PIV analysis - x = ' num2str(x)]);
set(a,'Fontsize',16);
a=xlabel('z');
set(a,'Fontsize',14);
a=legend('w', 'dw/dz','Location','NorthWest');
set(a,'Fontsize',14);
```



Sphere Motion in Fluid Flow



Equation of Motion – 2nd Order Differential Equation

$$M\frac{d^2x}{dt^2} = 1/2\rho C_d \pi R^2 \left(V - \frac{dx}{dt}\right)^2$$

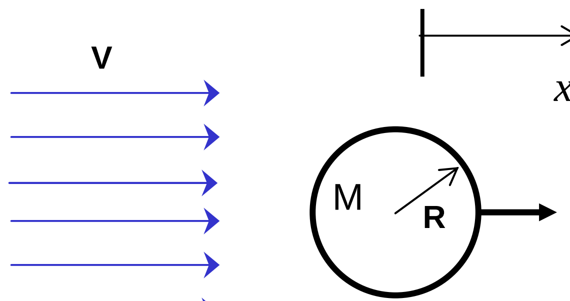
Rewite to 1st Order Differential Equations

$$\begin{aligned}\frac{dx}{dt} &= u \\ \frac{du}{dt} &= \frac{\rho C_d \pi R^2}{2M}(V^2 - 2uV + u^2)\end{aligned}$$

Euler' Method - Difference Equations

$$\begin{aligned}u_{i+1} &= u_i + \left(\frac{du}{dt}\right)_i \Delta t, \quad u(0) = 0 \\ x_{i+1} &= x_i + \left(\frac{du}{dt}\right)_i \Delta t, \quad x(0) = 0\end{aligned}$$

Sphere Motion in Fluid Flow
MATLAB Solutions



$$u = \frac{dx}{dt}$$

dudt.m

```
function [f] = dudt(t,u)
% u(1) = u
% u(2) = x
% f(2) = dx/dt = u
% f(1) = du/dt=rho*Cd*pi*r/(2m)*(v^2-2uv+u^2)
rho=1000;
Cd=1;
m=5;
r=0.05;
fac=rho*Cd*pi*r^2/(2*m);
v=1;

f(1)=fac*(v^2-2*u(1)+u(1)^2);
f(2)=u(1);
f=f';
```

sph_drag_2.m

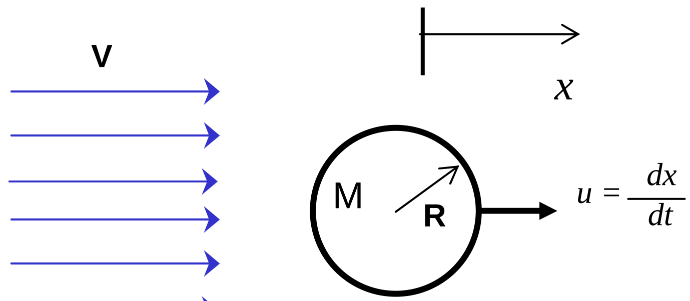
```
x=[0:0.1:10];
%step size
h=1.0;
% Euler's method, forward finite difference
t=[0:h:10];
N=length(t);
u_e=zeros(N,1);
x_e=zeros(N,1);
u_e(1)=0;
x_e(1)=0;
for n=2:N
    u_e(n)=u_e(n-1)+h*fac*(v^2-2*v*u_e(n-1)+u_e(n-1)^2);
    x_e(n)=x_e(n-1)+h*u_e(n-1);
end
% Runge Kutta
u0=[0 0]';
[tt,u]=ode45(@dudt,t,u0);

figure(1)
hold off
a=plot(t,u_e,'+b');
hold on
a=plot(tt,u(:,1),'.g');
a=plot(tt,abs(u(:,1)-u_e),'+r');
...
figure(2)
hold off
a=plot(t,x_e,'+b');
hold on
a=plot(tt,u(:,2),'.g');
a=plot(tt,abs(u(:,2)-x_e),'xr');
...
```

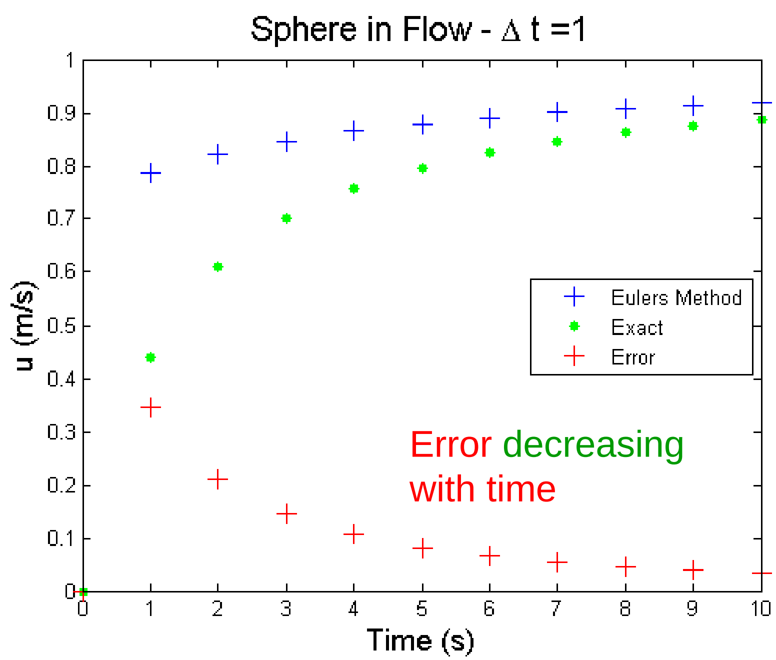
$$u_{i+1} = u_i + \left(\frac{du}{dt}\right)_i \Delta t, \quad u(0) = 0$$

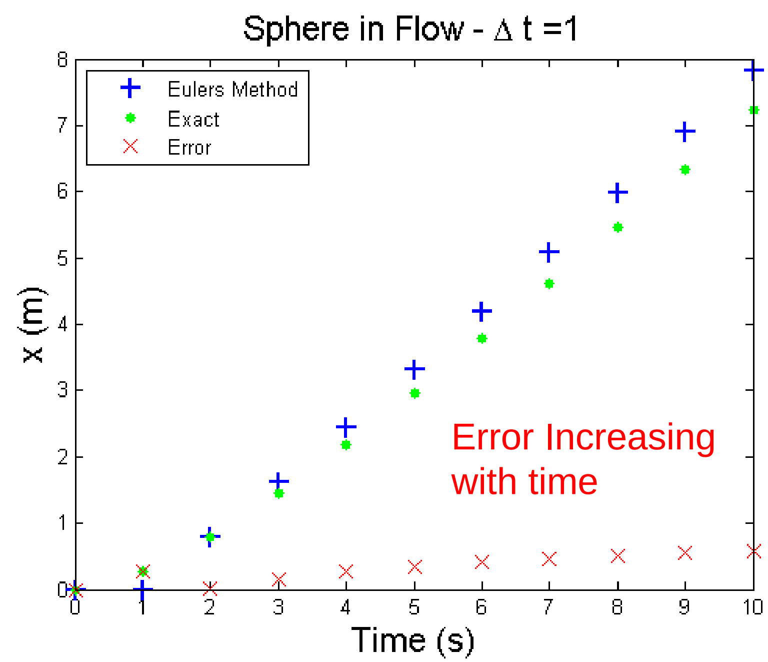
$$x_{i+1} = x_i + \left(\frac{du}{dt}\right)_i \Delta t, \quad x(0) = 0$$

Sphere Motion in Fluid Flow

Error Propagation



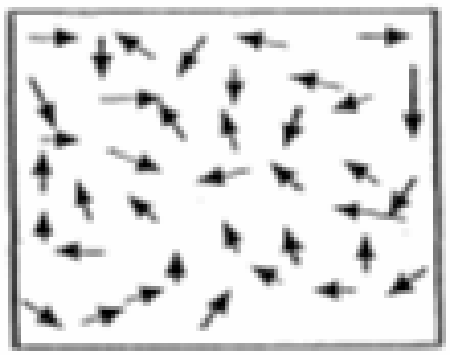




Incompressive Fluid Mechanics

Navier-Stokes Equation

V(x,y,z)



Fluid Velocity Field

$$\mathbf{V} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

Conservation of Mass

$$\begin{aligned} \mathrm{div}\mathbf{V} = \nabla \cdot \mathbf{V} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \end{aligned}$$

Dynamic Pressure $P = P_T + \rho g z$

Navier-Stokes Equation

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{V}$$

Density ρ

Kinematic viscosity ν

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}$$

Incompressive Fluid
Pressure Equation

Navier-Stokes Equation

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{V}$$

Conservation of Mass

$$\nabla \cdot \mathbf{V} = 0$$

Divergence of Navier-Stokes Equation

$$\mathrm{div}(\mathbf{V} \cdot \nabla) = -\frac{1}{\rho}\nabla^2 P$$

Dynamic Pressure

$$\nabla^2 P = -\rho\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + 2\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\right\}$$

More general than Bernoulli – Valid for unsteady and rotational flow

Incompressive Fluid
Vorticity Equation

Vorticity

$$\tilde{\omega} \equiv \mathrm{curl}\mathbf{V} \equiv \nabla \times \mathbf{V}$$

Navier-Stokes Equation

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{V}$$

curl of Navier-Stokes Equation

$$\frac{D\tilde{\omega}}{Dt} = -(\tilde{\omega} \cdot \nabla)\mathbf{V} + \nu \nabla^2 \tilde{\omega}$$

Inviscid Fluid Mechanics
Euler's Equation

Navier-Stokes Equation

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{V}$$

Inviscid Fluid

$$\nu = 0$$

Euler's Equations

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial y} \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial z}
\end{aligned}$$

Inviscid Fluid Mechanics

Bernoulli Theorems

Theorem 1

Irrotational Flow

$$\nabla \times \mathbf{V} = \mathbf{0}$$

Flow Potential

$$\mathbf{V} = \nabla \phi$$

Define

$$H = \frac{1}{2}|\mathbf{V}|^2 + \frac{P}{\rho}$$

$$\frac{\partial \phi}{\partial t} + H = 0$$

$$P_T = P - \rho g z$$

$$\boxed{\frac{\partial \phi}{\partial t} + \frac{1}{2}|\mathbf{V}|^2 + \frac{P_T}{\rho} + gz = 0}$$

Theorem 2

Steady Flow

Navier-Stokes Equation

$$\cancel{\frac{\partial \mathbf{V}}{\partial t}} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P$$

$$\mathbf{V} \times \tilde{\omega} = \nabla H$$

Along stream lines and vortex lines

$$\boxed{\begin{aligned} H &= \frac{1}{2}|\mathbf{V}|^2 + \frac{P}{\rho} \\ &= \frac{1}{2}|\mathbf{V}|^2 + \frac{P_T}{\rho} + gz = \text{const} \end{aligned}}$$

Potential Flows

Integral Equations

Irrotational Flow

$$\nabla \times \mathbf{V} = 0$$

Flow Potential

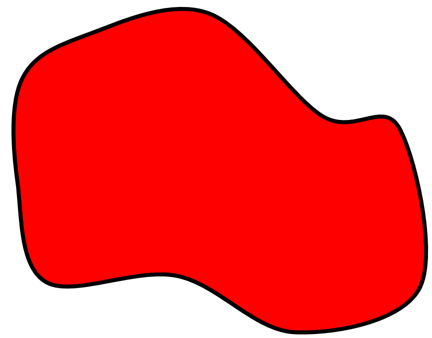
$$\mathbf{V} = \nabla \phi$$

Conservation of Mass

$$\begin{aligned} \nabla \cdot \mathbf{V} &= 0 \\ &\Rightarrow \\ \nabla \cdot (\nabla \phi) &= 0 \end{aligned}$$

Laplace Equation

$$\boxed{\nabla^2 \phi = 0}$$

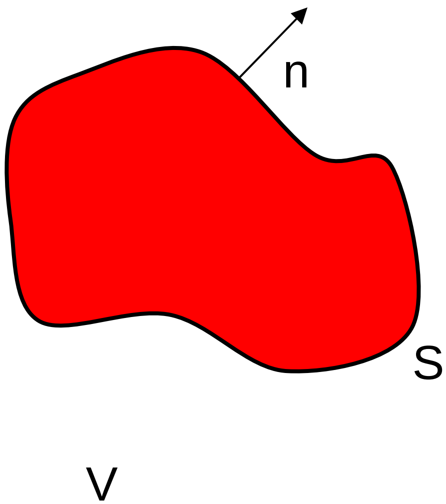


Mostly Potential Flow:

Only rotation at boundaries

Potential Flow

Boundary Integral Equations



Green's Theorem

$$\int_S \left[G(\mathbf{x},\mathbf{x}_0)\frac{\partial \phi(\mathbf{x}_0)}{\partial n} - \phi(\mathbf{x}_0)\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial n} \right] dS_0$$

$$= \int_V \left[\phi(\mathbf{x}_0)\nabla^2 G(\mathbf{x},\mathbf{x}_0) - G(\mathbf{x},\mathbf{x}_0)\nabla^2 \phi(\mathbf{x}_0) \right] dV_0$$

Green's Function

$$G(\mathbf{x},\mathbf{x}_0) = \frac{1}{r} = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \psi(\mathbf{x})$$

Homogeneous Solution

$$\nabla^2 \psi = 0$$

$$\nabla^2 G((\mathbf{x},\mathbf{x}_0) = -\delta(\mathbf{x}-\mathbf{x}_0)$$

Boundary Integral Equation

$$\phi(\mathbf{x}) = \int_S \left[G(\mathbf{x},\mathbf{x}_0)\frac{\partial \phi(\mathbf{x}_0)}{\partial n} - \phi(\mathbf{x}_0)\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial n} \right] dS_0 - \int_V \left[G(\mathbf{x},\mathbf{x}_0)\nabla^2 \phi(\mathbf{x}_0) \right] dV_0$$

Discretized Integral Equation

$$\sum_{j=0}^{N-1} A_{ij} w_j = B_i$$

Linear System of Equations

$$\overline{\overline{\mathbf{A}}}\mathbf{u} = \mathbf{b}$$

$\boxed{\text{Panel Methods}}$