Numerical Marine Hydrodynamics

- Partial Differential Equations
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    - Heat Equation
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      - Crank-Nicholson
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Partial Differential Equations

Quasi-linear PDE

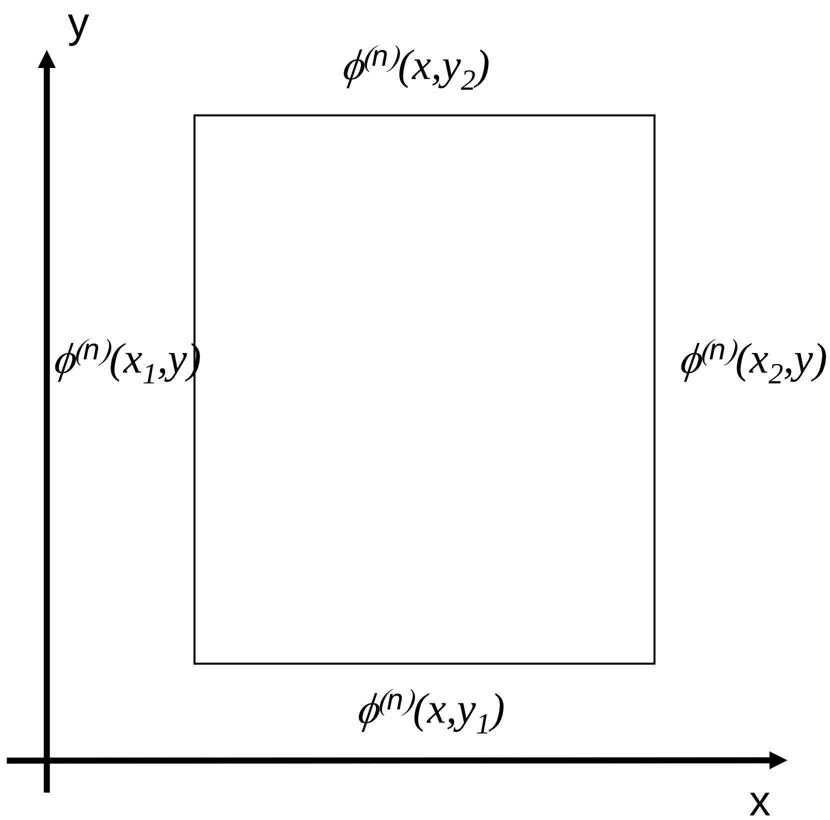
$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = F(x, y, \phi, \phi_x, \phi_y)$$

A,B and C Constants

$$B^2 - 4AC > 0 \quad \text{Hyperbolic}$$

$$B^2 - 4AC = 0 \quad \text{Parabolic}$$

$$B^2 - 4AC < 0 \quad \text{Elliptic}$$



Partial Differential Equations
Elliptic PDE

Laplace Operator

$$\nabla^2 \equiv u_{xx} + u_{yy}$$

Examples

$$\nabla^2 u = 0$$

Laplace Equation – Potential Flow

$$\nabla^2 u = g(x,y)$$

Poisson Equation
- Potential Flow with sources
- Heat flow in plate

$$\nabla^2 u + f(x,y)u = 0$$

Helmholtz equation – Vibration of plates

Partial Differential Equations
Elliptic PDEs

$$0 \le x \le a \;,\;\; 0 \le y \le b;$$

Equidistant Sampling

$$h = a/(n-1)$$
$$h = b/(m-1)$$

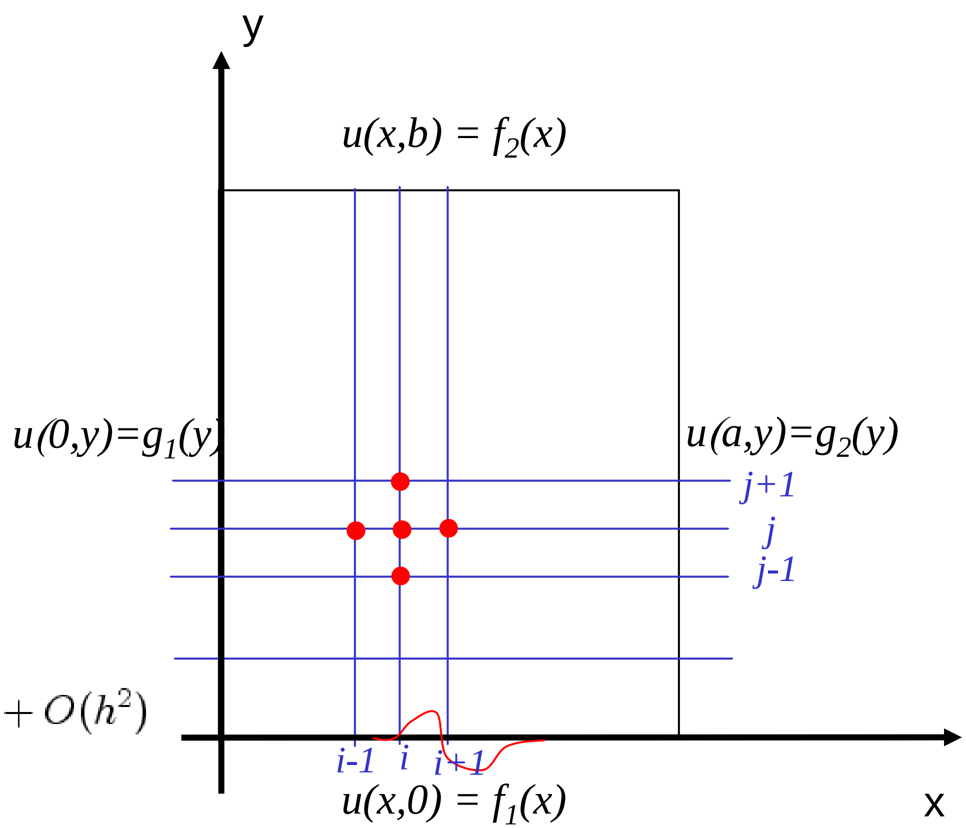
Discretization

$$x_i = (i-1)h, \; i = 1, \ldots, n$$
$$y_j = (j-1)h, \; j = 1, \ldots, m$$

Finite Differences

$$u_{xx}(x,t) = \frac{u(x_{i-1},y_j) - 2u(x_i,y_j) + u(x_{i+1},y_j)}{h^2} + O(h^2)$$

$$u_{yy}(x,t) = \frac{u(x_i,y_{j-1}) - 2u(x_i,y_j) + u(x_i,y_{j+1})}{h^2} + O(h^2)$$



Partial Differential Equations
Elliptic PDE

Laplace Equation

$$\nabla^2 u = \frac{u(x_{i-1}, y_j) + u(x_i, y_{j-1}) - 4u(x_i, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j+1})}{h^2} = 0$$

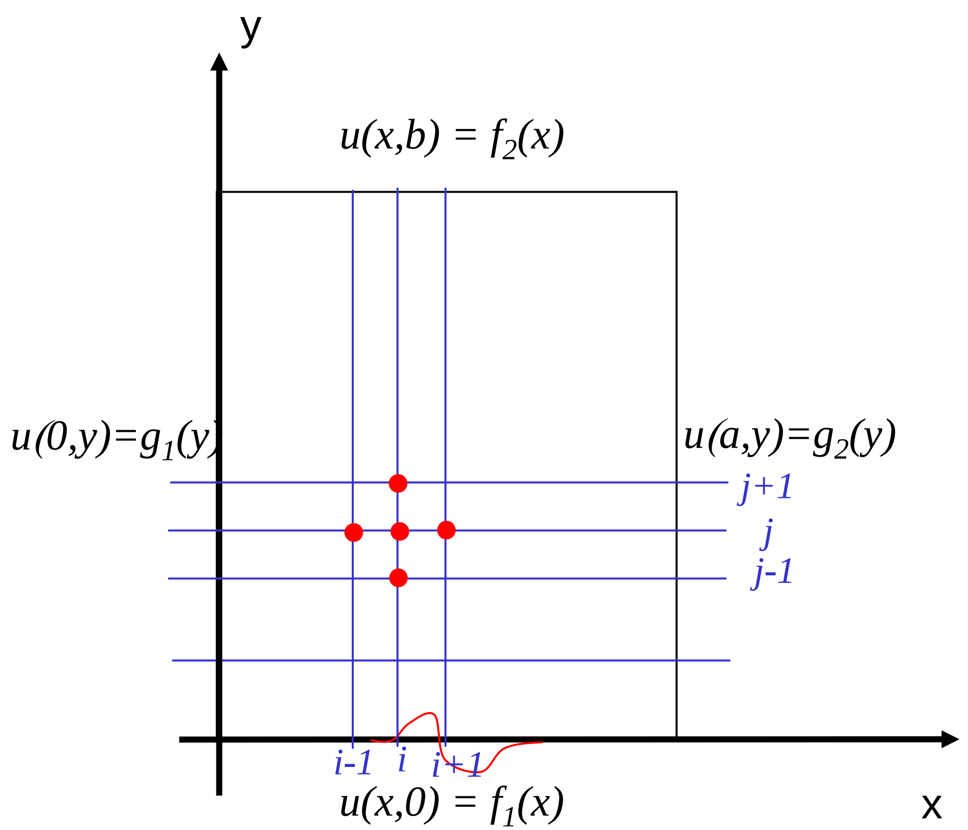
$$u_{i,j} = u(x_i, t_j)$$

Finite Difference Scheme

$$u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 0$$

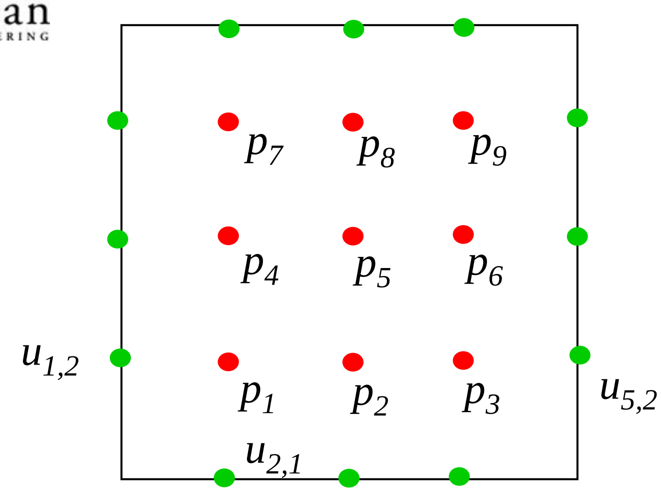
Boundary Conditions

$$\begin{aligned} u(x_1, y_j) &= u_{1,j} \;, \quad 2 \le j \le m-1 \\ u(x_n, y_j) &= u_{n,j} \;, \quad 2 \le j \le m-1 \\ u(x_i, y_1) &= u_{i,1} \;, \quad 2 \le j \le n-1 \\ u(x_i, y_n) &= u_{i,n} \;, \quad 2 \le j \le n-1 \end{aligned}$$



Global Solution Required

Elliptic PDEs
Laplace Equation



$$-4p_1 + p_2 + p_4 = -u_{2,1} - u_{1.2}$$

$$p_1 - 4p_2 + p_3 + p_5 = -u_{3,1}$$

$$p_2 - 4p_3 + p_6 = -u_{4,1} - u_{5,2}$$

$$p_1 - 4p_4 + p_5 + p_7 = -u_{1,3}$$

$$p_2 = p_4 - 4p_5 + p_6 + p_8 = 0$$

$$p_3 + p_5 - 4p_6 + p_9 = -u_{5,3}$$

$$p_4 - 4p_7 + p_8 = -u_{2,5} - u_{1,4}$$

$$p_5 + p_7 - 4p_8 + p_9 = -u_{3,5}$$

$$-u_{4,5} \quad p_6 + p_8 - 4p_9 = -u_{4,5} - u_{5,4}$$

Ellipticic PDEs
Derivative Boundary Conditions

Neumann Boundary Condition

$$\frac{\partial}{\partial N} u(x,y) = 0$$

$$\frac{\partial}{\partial x} u(x_n, y_j) = u_x(x_n, y_j) = 0$$

Liebman Finite Difference Scheme

$$u_{n+1,j} + u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} = 0$$

Derivative Finite Difference

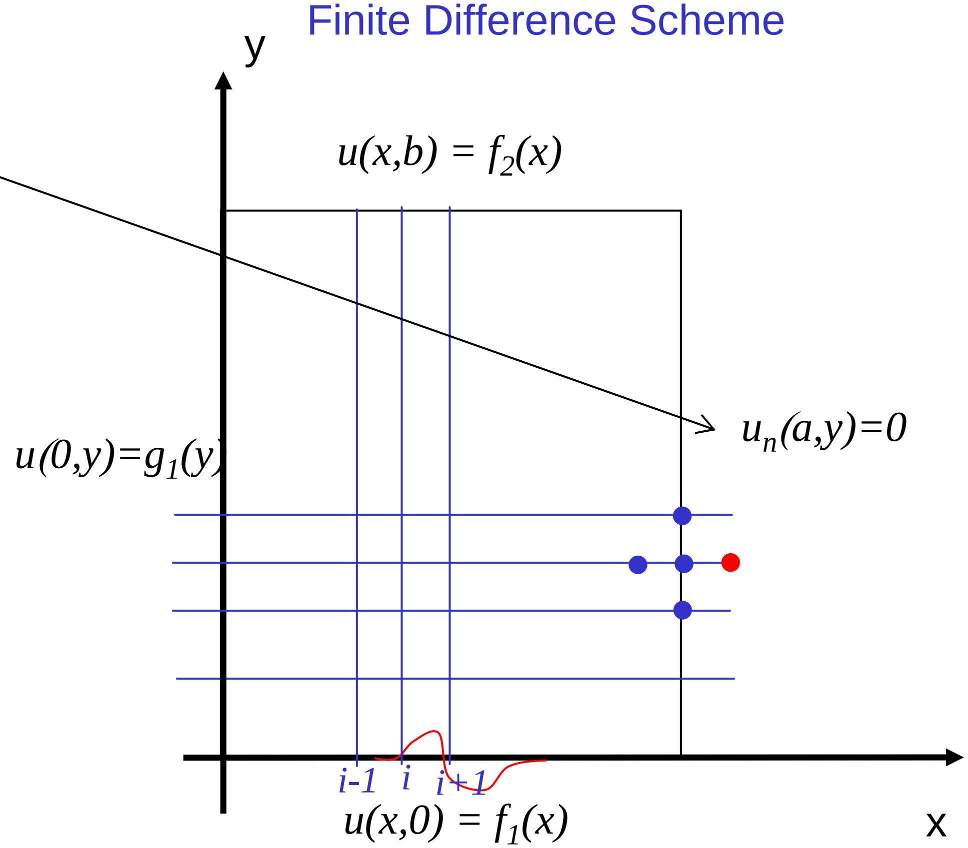
$$\frac{u_{n+1,j} - u_{n-1,j}}{2h} \simeq u_x(x_n, y_j)$$

$$\boxed{u_{n+1,j}} = u_{n-1,j} + 2h u_x(x_n, y_j) = u_{n-1,j}$$

Boundary Finite Difference Scheme

$$2u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} = 0$$

Finite Difference Scheme



Elliptic PDEs
Iterative Schemes

Finite Difference Scheme

$$u_{n+1,j} + u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} = 0$$

Liebman Iterative Scheme

$$u_{i,j}^{k+1} = u_{i,j}^{k} + r_{i,j}^{k}$$

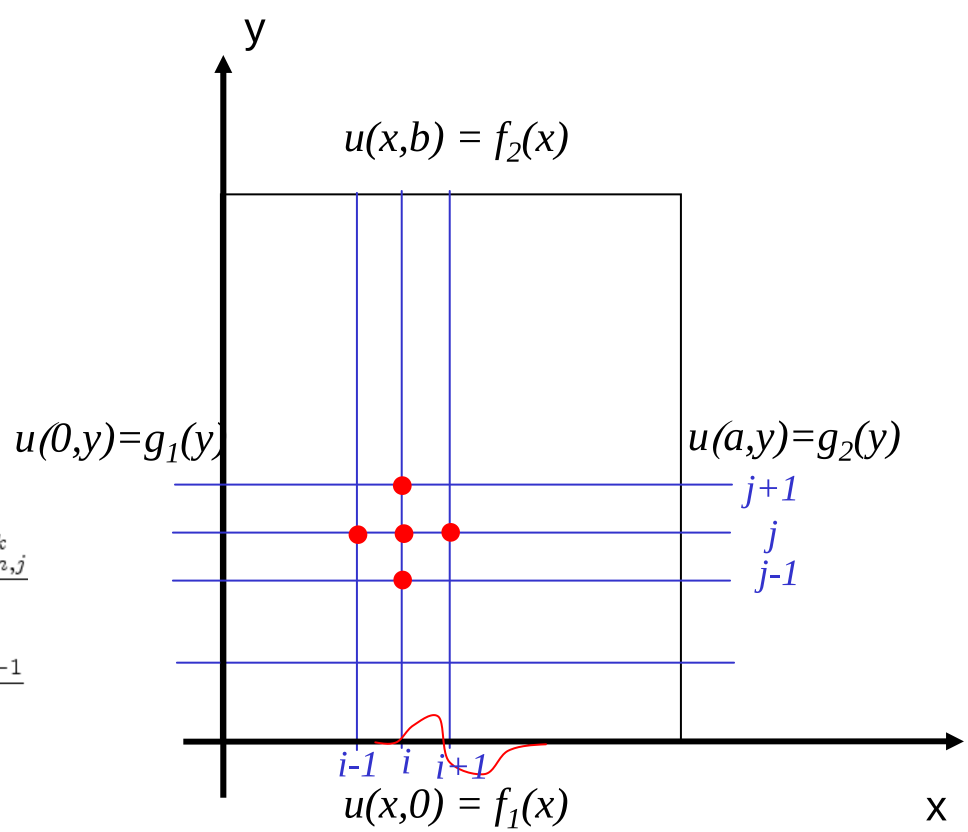
$$r_{i,j} = \frac{u_{n+1,j} + u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j}}{4}$$

SOR Iterative Scheme

$$\begin{aligned} u_{i,j}^{k+1} &= u_{i,j}^{k} + \omega r_{i,j}^{k} \\ &= u_{i,j}^{k} + \omega \frac{u_{n+1,j}^{k} + u_{n-1,j}^{k} + u_{n,j+1}^{k} + u_{n,j-1}^{k} - 4u_{n,j}^{k}}{4} \\ &= (1-\omega) u_{i,j}^{k} + \omega \frac{u_{n+1,j}^{k} + u_{n-1,j}^{k} + u_{n,j+1}^{k} + u_{n,j-1}^{k}}{4} \end{aligned}$$

Optimal SOR

$$\omega = \frac{4}{2 + \sqrt{4 - \left[\cos\left(\frac{\pi}{n-1}\right) + \cos\left(\frac{\pi}{m-1}\right)\right]^2}}$$



Laplace Equation
Flow in Duct

duct.m

```
Lx=1;
Ly=1;
N=10;
h=Lx/N;
M=floor(Ly/Lx*N);
niter=20;
eps=1e-6;

x=[0:h:Lx]';
y=[0:h:Ly];
f1x='4*x-4*x.^2';
%f1x='0'
f2x='0';
g1x='0';
g2x='0';
vxy='0';
f1=inline(f1x,'x');
f2=inline(f2x,'x');
g1=inline(g1x,'y');
g2=inline(g2x,'y');
vf=inline(vxy,'x','y');

n=length(x);
m=length(y);
u=zeros(n,m);
u(2:n-1,1)=f1(x(2:n-1));
u(2:n-1,m)=f2(x(2:n-1));
u(1,1:m)=g1(y);
u(n,1:m)=g2(y);
for i=1:n
    for j=1:m
        v(i,j) = vf(x(i),y(j));
    end
end
```

```
u_0=mean(u(1,:))+mean(u(n,:))+mean(u(:,1))+mean(u(:,m));
u(2:n-1,2:m-1)=u_0*ones(n-2,m-2);
omega=4/(2+sqrt(4-(cos(pi/(n-1))+cos(pi/(m-1)))^2))
for k=1:niter
    u_old=u;
    for i=2:n-1
        for j=2:m-1
            u(i,j)=(1-omega)*u(i,j)
 +omega*(u(i-1,j)+u(i+1,j)+u(i,j-1)+u(i,j+1)-h^2*v(i,j))/4;
        end
    end
    r=abs(u-u_old)/max(max(abs(u)));
    k,r
    if (max(max(r))<eps)
        break;
    end
end
figure(3)
surf(y,x,u);
shading interp;
a=ylabel('x');
set(a,'Fontsize',14);
a=xlabel('y');
set(a,'Fontsize',14);
a=title(['Poisson Equation - v = ' vxy]);
set(a,'Fontsize',16);
```

Poisson Equation

$$\nabla^2 u = g(x, y)$$

$$g_{i,j} = g(x_i, y_j)$$

SOR Iterative Scheme

$$\begin{aligned} u_{i,j}^{k+1} &= u_{i,j}^k + \omega r_{i,j}^k \\ &= u_{i,j}^k + \omega \frac{u_{n+1,j}^k + u_{n-1,j}^k + u_{n,j+1}^k + u_{n,j-1}^k - 4u_{n,j}^k - h^2 g_{i,j}}{4} \\ &= \boxed{(1-\omega)u_{i,j}^k + \omega \frac{u_{n+1,j}^k + u_{n-1,j}^k + u_{n,j+1}^k + u_{n,j-1}^k - h^2 g_{i,j}}{4}} \end{aligned}$$

Helmholtz Equation

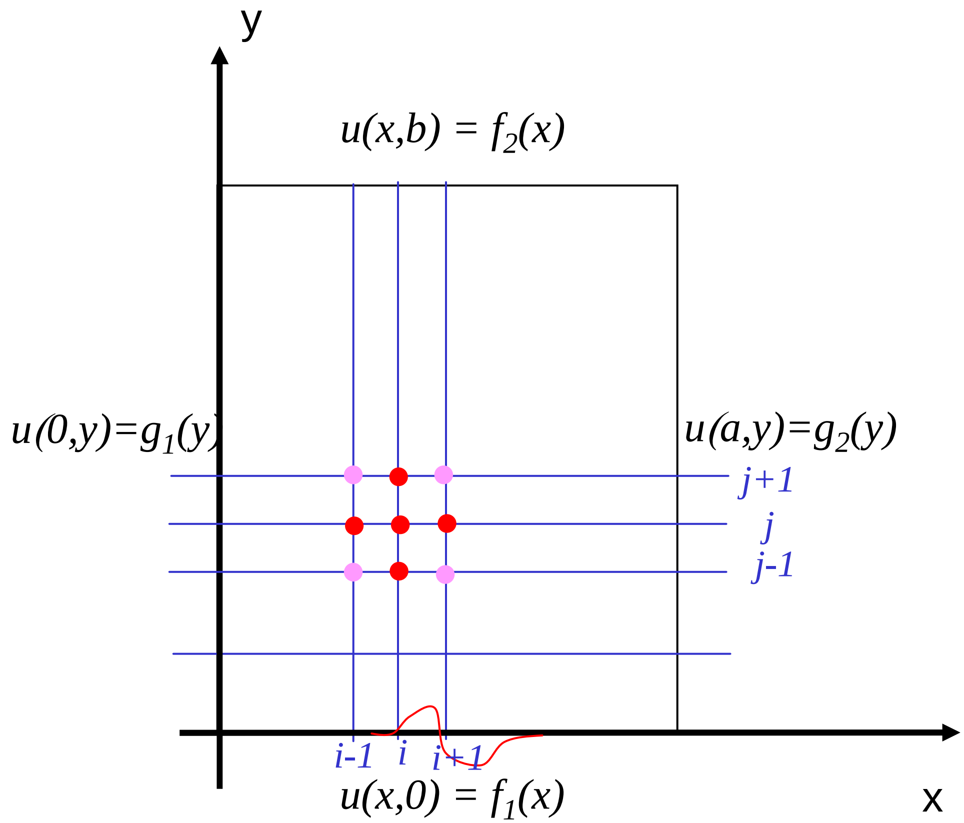
$$\nabla^2 u + f(x,y)u = g(x,y)$$

$$\begin{aligned} f_{i,j} &= f(x_i, y_j) \\ g_{i,j} &= g(x_i, y_j) \end{aligned}$$

SOR Iterative Scheme

$$\begin{aligned} u_{i,j}^{k+1} &= u_{i,j}^k + \omega r_{i,j}^k \\ &= u_{i,j}^k + \omega \frac{u_{n+1,j}^k + u_{n-1,j}^k + u_{n,j+1}^k + u_{n,j-1}^k - (4 - h^2 f_{i,j}) u_{n,j}^k - h^2 g_{i,j}}{4 - h^2 f_{i,j}} \\ &= \boxed{(1-\omega) u_{i,j}^k + \omega \frac{u_{n+1,j}^k + u_{n-1,j}^k + u_{n,j+1}^k + u_{n,j-1}^k - h^2 g_{i,j}}{4 - h^2 f_{i,j}}} \end{aligned}$$

Elliptic PDE's
Higher Order Finite Differences



$$
\nabla^2 u_{i,j} = \frac{1}{6h^2}[u_{i+1,j-1} + u_{i-1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}
$$

$$
+4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j+1} + 4u_{i,j-1} - 20u_{i,j}] + O(h^4)
$$

Elliptic PDEs
Irregular Boundaries

Dirichlet Boundary Conditions

$$\left(\frac{\partial u}{\partial x}\right)_{i-1,i} \simeq \frac{u_{i,j} - u_{i-1,j}}{\alpha_1 \Delta x}$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,i+1} \simeq \frac{u_{i+1,j} - u_{i,j}}{\alpha_2 \Delta x}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\ &= \frac{\left(\frac{\partial u}{\partial x}\right)_{i,i+1} - \left(\frac{\partial u}{\partial x}\right)}{\frac{\alpha_1 \Delta x + \alpha_2 \Delta x}{2}} \end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2}{\Delta x^2}\left[\frac{u_{i-1,j} - u_{i,j}}{\alpha_1(\alpha_1 + \alpha_2)} + \frac{u_{i+1,j} - u_{i,j}}{\alpha_2(\alpha_1 + \alpha_2)}\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2}{\Delta y^2}\left[\frac{u_{i,j-1} - u_{i,j}}{\beta_1(\beta_1 + \beta_2)} + \frac{u_{i,j+1} - u_{i,j}}{\beta_2(\beta_1 + \beta_2)}\right]$$

Elliptic PDEs
Irregular Boundaries

Neumann Boundary Conditions

$$\frac{\partial u}{\partial \eta} = \frac{u_1 - u_7}{L_{17}}$$

$$\begin{aligned} L_{78} &= \Delta x \tan\theta \\ L_{17} &= \Delta x / \cos\theta \end{aligned}$$

$$u_7 = u_8 + (u_6 - u_8)\frac{\Delta x \tan\theta}{\Delta y}$$

$$u_1 = \left(\frac{\Delta x}{\cos\theta}\right)\frac{\partial u}{\partial \eta} + u_6 \frac{\Delta x \tan\theta}{\Delta y} + u_8 \left(1 - \frac{\Delta x \tan\theta}{\Delta y}\right)$$

Elliptic PDEs
Internal Boundaries

Velocity and Stress Continuity

$$u^+ = u^-$$
$$\mu^+ \frac{\partial u^+}{\partial y} = \mu^- \frac{\partial u^-}{\partial y}$$

Derivative Finite Differences

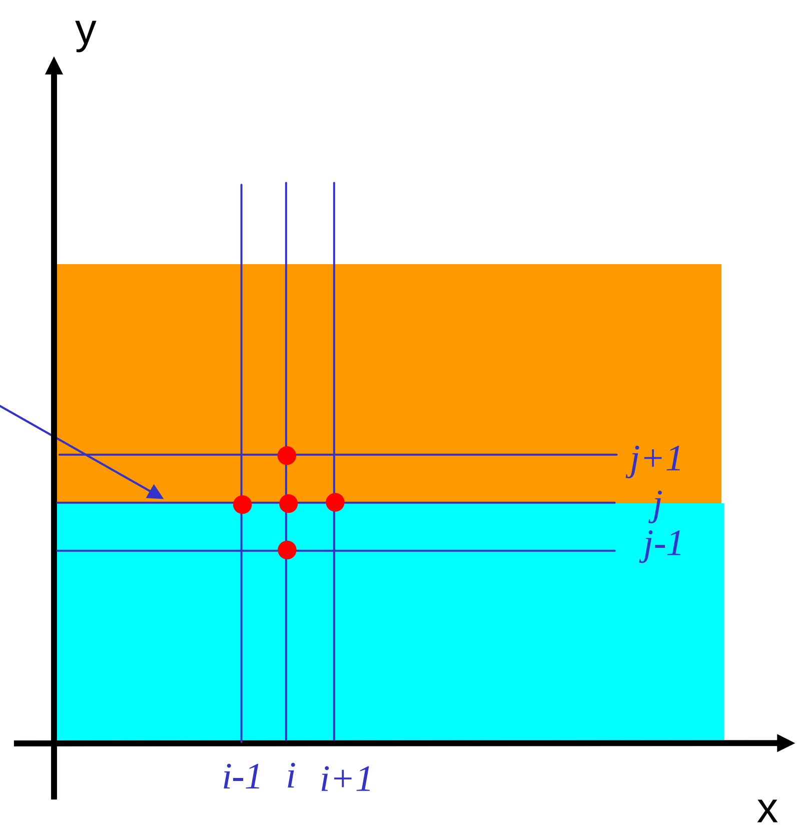
$$\mu^+ \frac{\partial u^+}{\partial y} = \mu^+ \frac{u_{i,j+1} - u_{i,j}}{h}$$
$$\mu^- \frac{\partial u^-}{\partial y} = \mu^- \frac{u_{i,j} - u_{i,j-1}}{h}$$

Finite Difference Equation

$$(\mu^- + \mu^+) u_i = \mu^+ u_{i,j+1} + \mu^- u_{i,j-1}$$

SOR Finite Difference Scheme

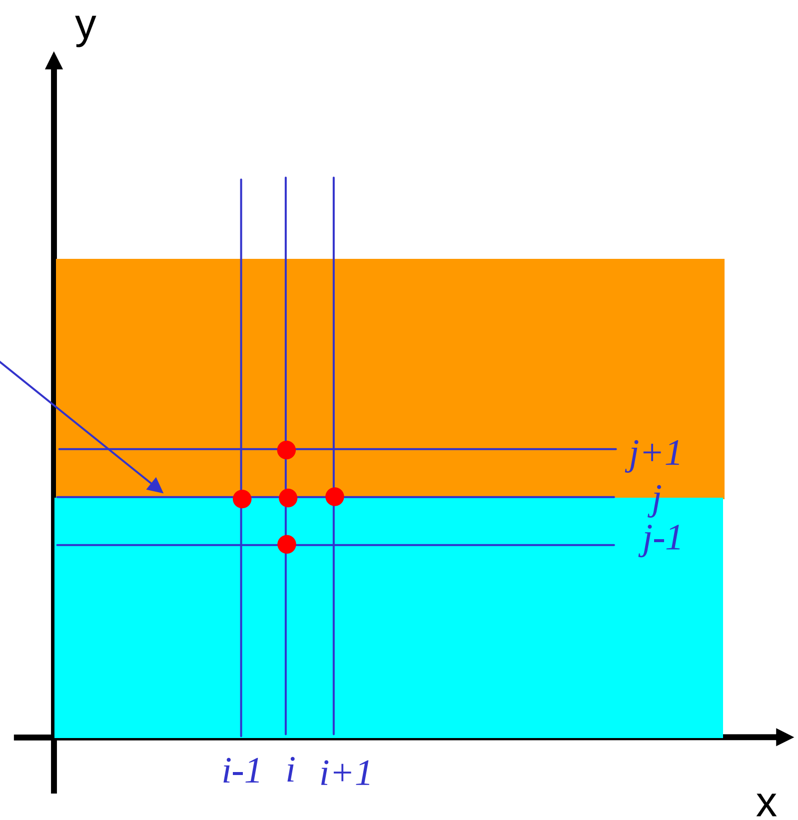
$$u_i^{k+1} = (1-\omega) u_i^k + \omega \frac{\mu^+ u_{i,j+1}^k + \mu^- u_{i.j-1}^k}{\mu^- + \mu^+}$$



Elliptic PDEs
Internal Boundaries – Higher Order

Velocity and Stress Continuity

$$u^+ = u^-$$
$$\mu^+ \frac{\partial u^+}{\partial y} = \mu^- \frac{\partial u^-}{\partial y}$$



Taylor Series

$$\begin{aligned} u_{i,j-1} &\simeq u_{i,j} - h u_y(x_i,y_j) + \frac{h^2}{2} u_{yy}(x_i,y_j) \\ &= u_{i,j} - h u_y(x_i,y_j) + \frac{h^2}{2}(g_{i,j} - u_x x(x_i,y_j)) \end{aligned}$$

$$\begin{aligned} u_{i,j+1} &\simeq u_{i,j} + h u_y(x_i,y_j) + \frac{h^2}{2} u_{yy}(x_i,y_j) \\ &= u_{i,j} + h u_y(x_i,y_j) + \frac{h^2}{2}(g_{i,j} - u_x x(x_i,y_j)) \end{aligned}$$

Derivative Finite Differences

$$\mu^+ \frac{\partial u^+}{\partial y} = \mu^+ \left[\frac{u_{i,j+1} - u_{i,j}}{h} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{2h} - \frac{h}{2} g_{i,j} \right]$$
$$\mu^- \frac{\partial u^-}{\partial y} = \mu^- \left[\frac{u_{i,j} - u_{i,j-1}}{h} - \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{2h} + \frac{h}{2} g_{i,j} \right]$$

Finite Difference Equation

$$\left[\frac{2(\mu^+ u_{i,j+1} + \mu^- u_{i.j-1})/(\mu^- + \mu^+) + u_{i+1,j} + u_{i-1,j} - 4u_{i,j} - h^2 g_{i,j}}{4} \right] = 0$$

SOR Finite Difference Scheme

$$u_i^{k+1} = (1-\omega) u_i^k + \omega \left[\frac{2(\mu^+ u^k_{i,j+1} + \mu^- u^k_{i.j-1})/(\mu^- + \mu^+) + u^k_{i+1,j} + u^k_{i-1,j} - h^2 g_{i,j}}{4} \right]$$