Potential Flow Formulation

Velocity $\vec{V} = \nabla\phi$

Governing Equations for P-Flow

(a) Continuity $\boxed{\nabla^2\phi = 0}$

(b) Bernoulli for P-Flow (steady or unsteady) $\boxed{p = -\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + gy\right) + C(t)}$

Boundary Conditions for P-Flow

Types of Boundary Conditions:

(c) Kinematic Boundary Conditions - specify the flow velocity $\vec{v}$ at boundaries. $\boxed{\frac{\partial\phi}{\partial n} = U_n}$

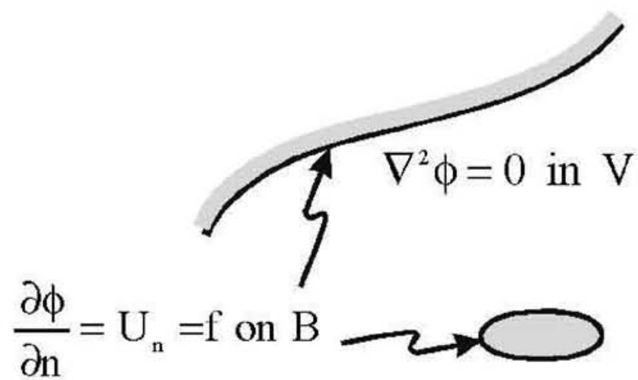
(d) Dynamic Boundary Conditions - specify force $\vec{F}$ or pressure p at flow boundary.

$$\boxed{p = -\rho\left(\phi_t + \frac{1}{2}(\nabla\phi)^2 + gy\right) + C(t) \ \ \text{(prescribed)}}$$

Linear Superposition for Potential Flow

In the **absence of dynamic boundary conditions**, the potential flow boundary value problem is **linear**.

- Potential function ϕ.



Linear Superposition: if $\phi_1, \phi_2, \ldots$ are harmonic functions, i.e., $\nabla^2\phi_i = 0$, then $\phi = \sum \alpha_i \phi_i$, where α_i are constants, are also harmonic, and is the solution for the boundary value problem provided the kinematic boundary conditions are satisfied, i.e.,

$$\frac{\partial \phi}{\partial n} = \frac{\partial}{\partial n}\left(\alpha_1\phi_1 + \alpha_2\phi_2 + \ldots\right) = U_n \quad \text{on} \quad \text{B}.$$

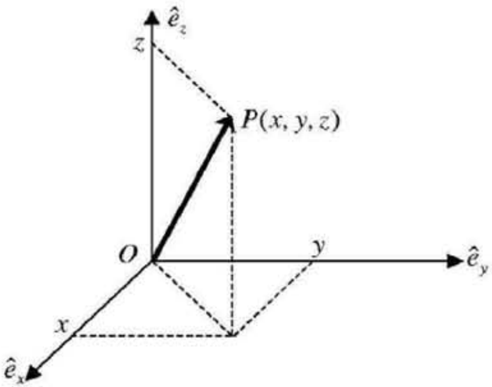
The key is to combine known solution of the Laplace equation in such a way as to satisfy the kinematic boundary conditions (KBC).

Laplace Equation in Different Coordinate Systems

Cartesian (x, y, z)

$$\vec{v} = \left(\overset{\hat{i}}{u}, \overset{\hat{j}}{v}, \overset{\hat{k}}{w}\right) = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$



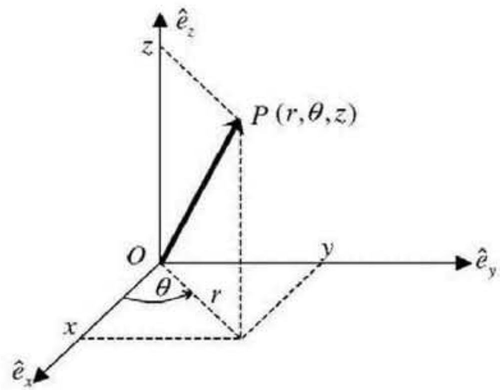
Cylindrical (r, θ, z)

$$r^2 = x^2 + y^2,$$
$$\theta = \tan^{-1}(y/x)$$

$$\vec{v} = \left(\overset{\hat{e}_r}{v_r}, \overset{\hat{e}_\theta}{v_\theta}, \overset{\hat{e}_z}{v_z}\right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla^2\phi = \underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} \Leftrightarrow$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$



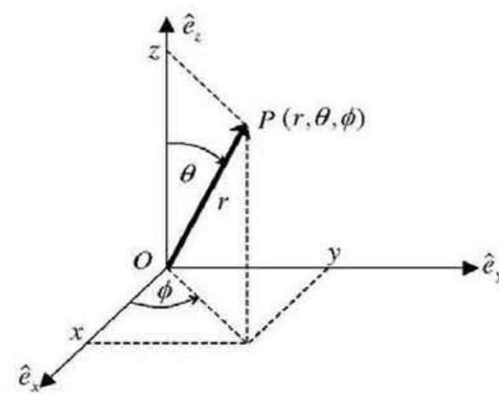
Spherical (r, θ, φ)

$$r^2 = x^2 + y^2 + z^2,$$
$$\theta = \cos^{-1}(z/r) \Leftrightarrow z = r(\cos\theta)$$
$$\varphi = \tan^{-1}(y/x)$$

$$\vec{v} = \nabla\phi = \left(\overset{\hat{e}_r}{v_r}, \overset{\hat{e}_\theta}{v_\theta}, \overset{\hat{e}_\varphi}{v_\varphi}\right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r(\sin\theta)}\frac{\partial\phi}{\partial\varphi}\right)$$

$$\nabla^2\phi = \underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} \Leftrightarrow$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$



Simple Potential Flows

1. **Uniform Stream** $\nabla^2(ax+by+cz+d)=0$

1D: $\phi = Ux +$ constant $\psi = Uy +$ constant; $\vec{v}=(U,0,0)$

2D: $\phi = Ux + Vy +$ constant $\psi = Uy - Vx +$ constant; $\vec{v}=(U,V,0)$

3D: $\phi = Ux + Vy + Wz +$ constant $\vec{v}=(U,V,W)$

2. **Source (sink) flow**

2D, Polar coordinates

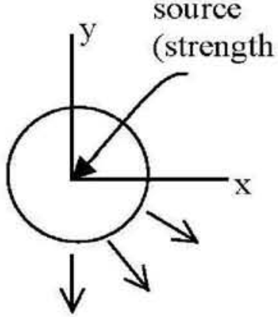
$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \text{ with } r=\sqrt{x^2+y^2}$$

An axisymmetric solution: $\phi = a\ln r + b$. Verify that it satisfies $\nabla^2\phi = 0$, except at $r=\sqrt{x^2+y^2}=0$. Therefor, $r=0$ must be excluded from the flow.

Define 2D source of strength m at $r=0$:

$$\phi = \frac{m}{2\pi}\ln r$$

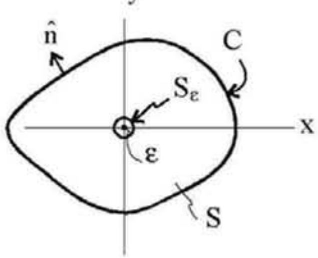
$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{e}_r = \frac{m}{2\pi r}\hat{e}_r \Longleftrightarrow v_r = \frac{m}{2\pi r},\ v_\theta = 0$$



Net outward volume flux is

$$\oint_C \vec{v}\cdot\hat{n}ds = \iint_S \nabla\cdot\vec{v}ds = \iint_{S_\varepsilon}\nabla\cdot\vec{v}ds$$

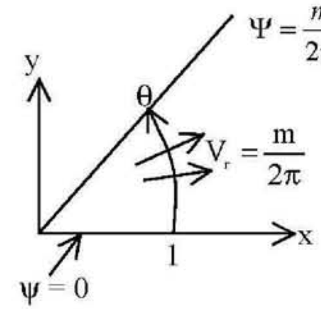
$$\oint_{C_\varepsilon} \vec{v}\cdot\hat{n}ds = \int_0^{2\pi} \underbrace{v_r}_{\frac{m}{2\pi r_\varepsilon}} r_\varepsilon d\theta = \underbrace{m}_{\text{source strength}}$$



If $m<0 \Rightarrow$ sink. Source m at (x_0,y_0):

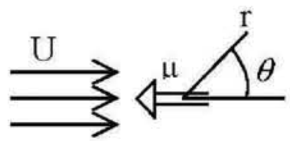
$$\phi = \frac{m}{2\pi}\ln\sqrt{(x-x_0)^2+(y-y_0)^2}$$

$$\phi = \frac{m}{2\pi}\ln r \text{ (Potential function)} \longleftrightarrow \psi = \frac{m}{2\pi}\theta \text{ (Stream function)}$$



Simple Potential Flows

7. **Stream + Dipole: circles and spheres**



2D: $\phi = Ux + \frac{\mu x}{2\pi r^2} \underset{x = r\cos\theta}{\overset{\uparrow}{=}} \cos\theta\left(Ur + \frac{\mu}{2\pi r}\right)$

The radial velocity is then

$$u_r = \frac{\partial \phi}{\partial r} = \cos\theta\left(U - \frac{\mu}{2\pi r^2}\right).$$

Setting the radial velocity $v_r = 0$ on $r = a$ we obtain $a = \sqrt{\frac{\mu}{2\pi U}}$. This is the K.B.C. for a stationary circle of radius a. Therefore, for

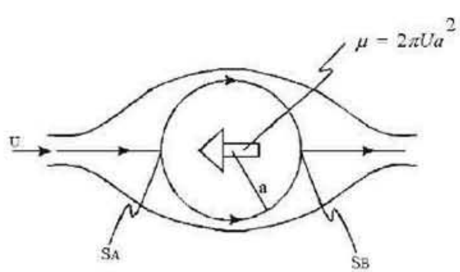
$$\mu = 2\pi U a^2$$

the potential

$$\phi = \cos\theta\left(Ur + \frac{\mu}{2\pi r}\right)$$

is the solution to ideal flow past a circle of radius a.

- *Flow past a circle* (U, a).



$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right)$

$V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right)$

$$V_\theta\big|_{r=a} = -2U\sin\theta \begin{cases} = 0 \text{ at } \theta = 0, \pi & \text{— stagnation points} \\ = \mp 2U \text{ at } \theta = \frac{\pi}{2}, \frac{3\pi}{2} & \text{— maximum tangential velocity} \end{cases}$$

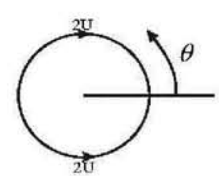
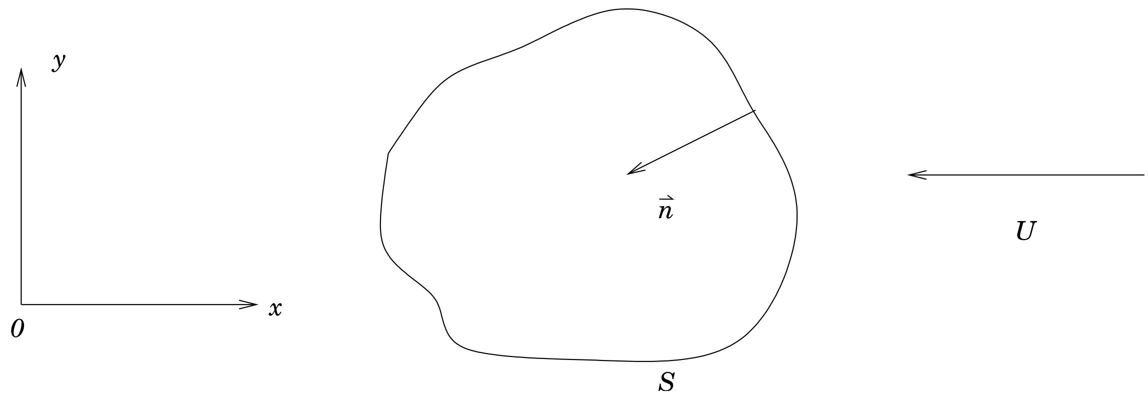


Illustration of the points where the flow reaches maximum speed around the circle.

2.29 - Numerical Fluid Mechanics Spring 2007

Lecture 22 - Example: Source Method

Dr.Yuming Liu



Boundary-value problem:

$$\Phi(x,y) = -Ux + \varphi(x,y)$$

$$\text{On } S\text{: } \Phi_n = 0 \quad \longrightarrow \quad \varphi_n = Un_x \tag{0}$$

$$\text{In the fluid: } \nabla^2\Phi = 0 \quad \longrightarrow \quad \nabla^2\varphi = 0$$

Source formulation:

$$\varphi(x,y) = \int_S \delta(\vec{\xi})G(\vec{x};\vec{\xi})ds(\vec{\xi}) \tag{1}$$

δ : unknown source distribution on S

$G(\vec{x};\vec{\xi}) = -ln|\vec{x}-\vec{\xi}|$: 2D Rankine source Green function

From (1), we have:

$$\frac{\partial\varphi(x,y)}{\partial n(\vec{x})} = \int_S \delta(\vec{\xi})\frac{\partial}{\partial n(\vec{x})}G(\vec{x};\vec{\xi})ds(\vec{\xi}) \tag{2}$$

Substituting (0) into (2), we obtain:

$$\int_S \delta(\vec{\xi})\frac{\partial}{\partial n(\vec{x})}G(\vec{x};\vec{\xi})ds(\vec{\xi}) = Un_x \qquad \text{for } \vec{x} \in S \tag{3}$$

To solve (3) numerically for $\delta(\vec{\xi})$, we apply constant panel method:

$$\begin{aligned} &\sum_{j=1}^{N}\int_{S_j}\delta(\vec{\xi})\frac{\partial}{\partial n(\vec{x})}G(\vec{x};\vec{\xi})ds(\vec{\xi}) = Un_x \\ \longrightarrow &\sum_{j=1}^{N}\delta_j\int_{S_j}\frac{\partial}{\partial n(\vec{x})}G(\vec{x};\vec{\xi})ds(\vec{\xi}) = Un_x \end{aligned} \tag{4}$$

For $\vec{x}=\vec{x}_i$ and with $\vec{x}_i$ being the position of the centroid of the $i-$th panel and δ_j the source strength of the $j-$th panel, (4) becomes:

$$\begin{aligned} &\sum_{j=1}^{N}\delta_j\underbrace{\int_{S_j}\frac{\partial}{\partial n(\vec{x}_i)}G(\vec{x}_i;\vec{\xi})ds(\vec{\xi})}_{A_{ij}} = \underbrace{Un_x(\vec{x}_i)}_{B_i} \\ \Longrightarrow\quad &\boxed{\sum_{j=1}^{N}\delta_j A_{ij} = B_i}\qquad i=1,2,\cdots N \\ \longrightarrow\quad &[A]\{\delta\}=\{B\} \end{aligned}$$

$$A_{ij} = \int_{S_j}\frac{\partial}{\partial n(\vec{x}_i)}G(\vec{x}_i;\vec{\xi})ds(\vec{\xi}) = n_x(\vec{x}_i)\cdot\int_{S_j}G_x(\vec{x}_i;\vec{\xi})ds(\vec{\xi}) + n_y(\vec{x}_i)\cdot\int_{S_j}G_y(\vec{x}_i;\vec{\xi})ds(\vec{\xi})$$

After $\{\delta\}$ is obtained:

$$\begin{aligned} \varphi(\vec{x}_i) &= \int_S\delta(\vec{\xi})G(\vec{x}_i;\vec{\xi})ds(\vec{\xi}) \\ &= \sum_{j=1}^{N}\int_{S_j}\delta(\vec{\xi})G(\vec{x}_i;\vec{\xi})ds(\vec{\xi}) \\ &= \sum_{j=1}^{N}\delta_j\int_{S_j}G(\vec{x}_i;\vec{\xi})ds(\vec{\xi}) \end{aligned} \tag{5}$$

Tangential velocity:

$$\begin{aligned} \varphi_\tau(\vec{x}_i) &= \frac{\partial}{\partial\tau}\varphi(\vec{x}_i) = \int_S\delta(\vec{\xi})\frac{\partial}{\partial\tau}G(\vec{x}_i;\vec{\xi})ds(\vec{\xi}) \\ &= \int_S\delta(\vec{\xi})\left[\tau_x G_x(\vec{x}_i;\vec{\xi}) + \tau_y G_y(\vec{x}_i;\vec{\xi})\right]ds(\vec{\xi}) \\ &= \sum_{j=1}^{N}\delta(\vec{\xi})\left[\tau_x(\vec{x}_i)\int_{S_j}G_x(\vec{x}_i;\vec{\xi})ds(\vec{\xi}) + \tau_y(\vec{x}_i)\int_{S_j}G_y(\vec{x}_i;\vec{\xi})ds(\vec{\xi})\right] \end{aligned} \tag{6}$$

Total Tangential velocity:

$$\Phi_\tau(\vec{x}_i) = -U\tau_x(\vec{x}_i) + \varphi_\tau(\vec{x}_i)$$

Pressure:

$$\begin{aligned}
\frac{P - \rho_\infty}{\rho} &= -\left[\cancelto{0}{\frac{\partial \Phi}{\partial t}} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi\right] + \frac{1}{2}U^2 \\
&= -\frac{1}{2}\left[\cancelto{0}{\Phi_n^2} + \Phi_\tau^2\right] + \frac{1}{2}U^2 \\
\therefore (P - \rho_\infty) &= -\frac{\rho}{2}\left[\Phi_\tau^2 - U^2\right]
\end{aligned}$$

Analytic Solution: (for a circular cylinder of radius R)

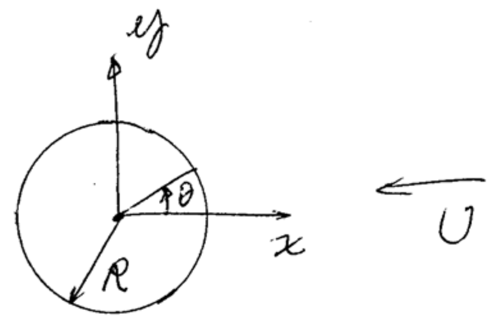
$$\begin{aligned}
&\varphi(\vec{x}) = -Ux && \text{for } \vec{x} \text{ on } S \\
&\Phi_\tau(\vec{x}) = 2U\sin\theta && \text{for } \vec{x} \text{ on } S \\
&C_\rho = \frac{P - \rho_\infty}{\frac{1}{2}\rho U^2} = 1 - 4\sin^2\theta
\end{aligned}$$

Cylinder.dat

Radius of the cylinder R=2. m.

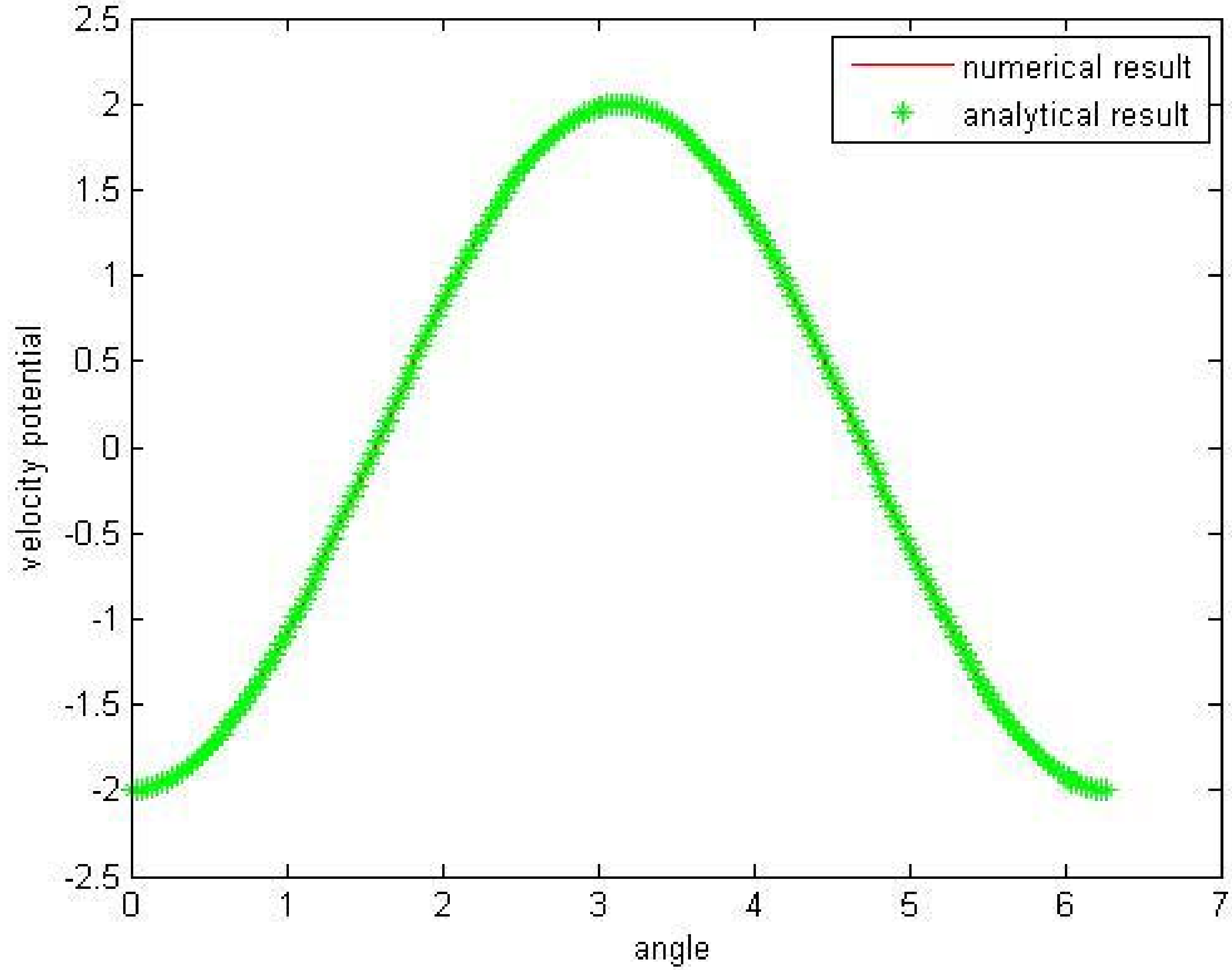
The center of the cylinder at x=0, y=0

Uniform flow speed U = 1 m/s

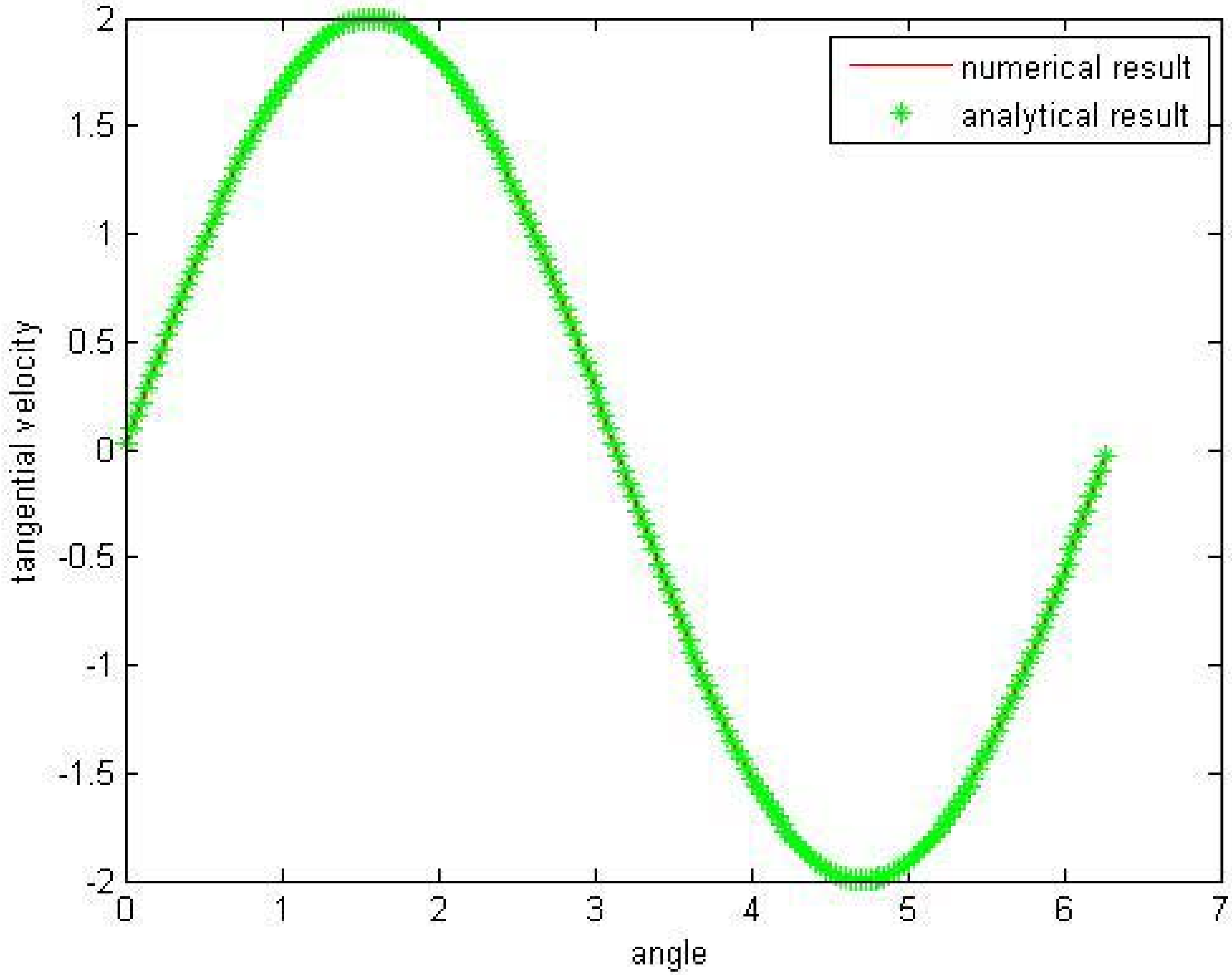


X	y
50	
2.0000	0.0000
1.9842	0.2507
1.9372	0.4974
1.8596	0.7362
1.7526	0.9635
1.6180	1.1756
1.4579	1.3691
1.2748	1.5410
1.0717	1.6887
0.8516	1.8097
0.6180	1.9021
0.3748	1.9646
0.1256	1.9961
-0.1256	1.9961
-0.3748	1.9646
-0.6180	1.9021
-0.8516	1.8097
-1.0717	1.6887
-1.2748	1.5410
-1.4579	1.3691
-1.6180	1.1756
-1.7526	0.9635
-1.8596	0.7362
-1.9372	0.4974
-1.9842	0.2507
-2.0000	0.0000
-1.9842	-0.2507
-1.9372	-0.4974
-1.8596	-0.7362
-1.7526	-0.9635
-1.6180	-1.1756
-1.4579	-1.3691
-1.2748	-1.5410
-1.0717	-1.6887
-0.8516	-1.8097
-0.6180	-1.9021
-0.3748	-1.9646
-0.1256	-1.9961
0.1256	-1.9961
0.3748	-1.9646
0.6180	-1.9021
0.8516	-1.8097
1.0717	-1.6887
1.2748	-1.5410
1.4579	-1.3691
1.6180	-1.1756
1.7526	-0.9635
1.8596	-0.7362
1.9372	-0.4974
1.9842	-0.2507
2.0000	-0.0000

Comparison Between Numerical Solution and Analytic solution (NPAN=200 panels)



Comparison Between Numerical Solution and Analytic solution (NPAN=200 panels)



Comparison Between Numerical Solution and Analytic solution (NPAN=200 panels)