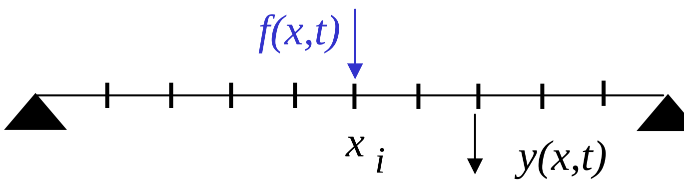
Introduction to Numerical Analysis for Engineers

- Systems of Linear Equations
  - Cramer's Rule
  - Gaussian Elimination
    - Numerical implementation
    - Numerical stability
      - Partial Pivoting
      - Equilibration
      - Full Pivoting
    - Multiple right hand sides
    - Computation count
    - LU factorization
    - Error Analysis for Linear Systems
      - Condition Number
    - Special Matrices
  - Iterative Methods
    - Jacobi's method
    - Gauss-Seidel iteration
    - Convergence

Linear Systems of Equations
Tri-diagonal Systems

Forced Vibration of a String



Harmonic excitation

$$f(x,t) = f(x)\cos(\omega t)$$

Differential Equation

$$\frac{d^2y}{dx^2} + k^2 y = f(x)$$

Boundary Conditions

$$y(0) = 0\,, \quad y(L) = 0$$

Finite Difference

$$\left.\frac{d^2y}{dx^2}\right|_{x_i} \simeq \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

Discrete Difference Equations

$$y_{i-1} + \left((kh)^2 - 2\right) y_i - y_{i+1} = f(x_i)h^2$$

Matrix Form

$$\begin{bmatrix} (kh)^2-2 & 1 & . & . & . \; . & 0 \\ 1 & (kh)^2-2 & 1 & & & . \\ . & & . & . & & . \\ . & & 1 & (kh)^2-2 & 1 & . \\ . & & & . & . & . \\ 0 & . & . & . & . \; 1 & (kh)^2-2 \end{bmatrix} \overline{\mathbf{x}} = \begin{Bmatrix} f(x_1)h^2 \\ . \\ . \\ f(x_i)h^2 \\ . \\ . \\ f(x_n)h^2 \end{Bmatrix}$$

Tridiagonal Matrix

$kh < 1$ Symmetric, positive definite: No pivoting needed

Linear Systems of Equations
Tri-diagonal Systems

General Tri-diagonal Systems

$$\begin{bmatrix} a_1 & c_1 & \cdot & \cdot & \cdot & \cdot & 0 \\ b_2 & a_2 & c_2 & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & b_i & a_i & c_i & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & b_n & a_n \end{bmatrix} \bar{\mathbf{x}} = \begin{Bmatrix} f_1 \\ \cdot \\ \cdot \\ f_i \\ \cdot \\ \cdot \\ f_n \end{Bmatrix}$$

LU Factorization

$$\overline{\overline{\mathbf{A}}} = \overline{\overline{\mathbf{L}}}\,\overline{\overline{\mathbf{U}}}$$

$$\overline{\overline{\mathbf{L}}} = \begin{bmatrix} 1 & & \cdot & \cdot & \cdot & \cdot & 0 \\ \beta_2 & 1 & & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & \beta_i & 1 & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \beta_n & 1 \end{bmatrix}$$

$$\overline{\overline{\mathbf{U}}} = \begin{bmatrix} \alpha_1 & c_1 & \cdot & \cdot & \cdot & \cdot & 0 \\ & \alpha_2 & c_2 & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \alpha_i & c_i & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & & \alpha_n \end{bmatrix}$$

$$\begin{aligned} \overline{\overline{\mathbf{L}}}\bar{\mathbf{y}} &= \bar{\mathbf{f}} \\ \overline{\overline{\mathbf{U}}}\bar{\mathbf{x}} &= \bar{\mathbf{y}} \end{aligned}$$

Linear Systems of Equations
Tri-diagonal Systems

LU Factorization

$$\overline{\overline{\mathbf{L}}} = \begin{bmatrix} 1 & & \cdot & \cdot & \cdot & \cdot & 0 \\ \beta_2 & 1 & & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & \beta_i & 1 & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \beta_n & 1 \end{bmatrix}$$

$$\overline{\overline{\mathbf{U}}} = \begin{bmatrix} \alpha_1 & c_1 & \cdot & \cdot & \cdot & \cdot & 0 \\ & \alpha_2 & c_2 & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \alpha_i & c_i & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & & \alpha_n \end{bmatrix}$$

Reduction

$$\alpha_1 = a_1$$

$$\beta_k = \frac{b_k}{\alpha_{k-1}} \;, \quad \alpha_k = a_k - \beta_k c_{k-1} \;, \; k = 2, 3, \ldots n$$

Forward Substitution

$$y_1 = f_1 \;, \quad y_i = f_i - \beta_i y_{i-1} \;, \; i = 2, 3, \ldots n$$

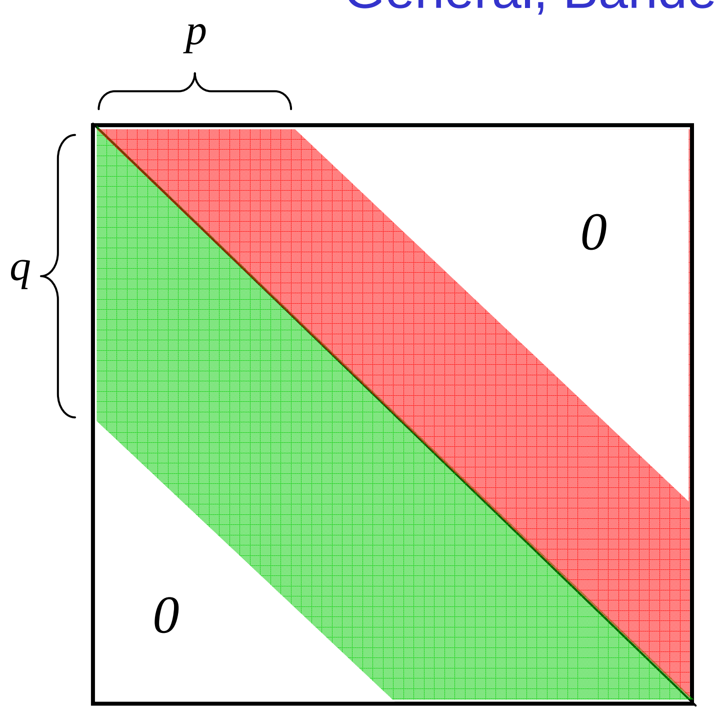
Back Substitution

$$x_n = \frac{y_n}{\alpha_n} \;, \quad x_i = \frac{y_i - c_i x_{i+1}}{\alpha_1} \;, \; i = n-1, \ldots 1$$

LU Factorization:	2*(n-1) operations
Forward substitution:	n-1 operations
Back substitution:	n-1 operations
Total:	4(n-1) ~ O(n) operations

Linear Systems of Equations
Special Matrices

General, Banded Coefficient Matrix



$$\left.\begin{array}{l} j > i + p \\ i > j + q \end{array}\right\} a_{ij} = 0$$

p super-diagonals
q sub-diagonals
$w = p+q+1$ bandwidth

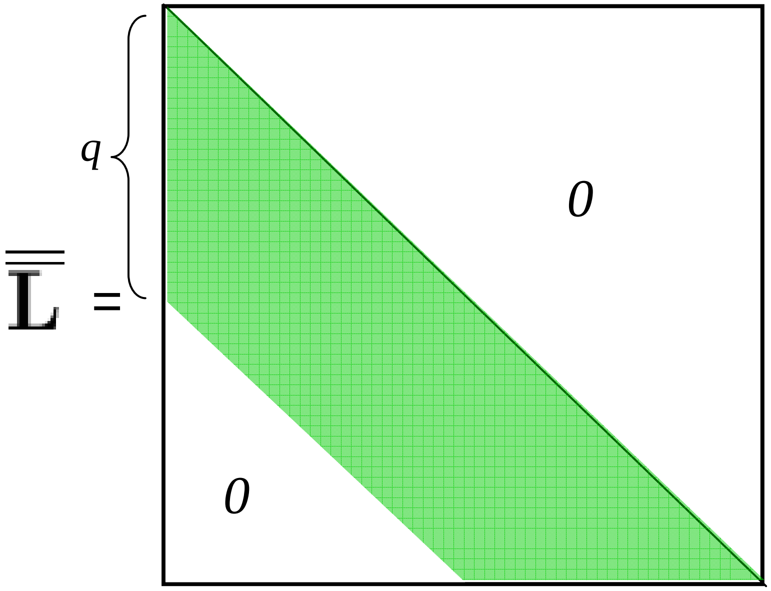
Banded Symmetric Matrix

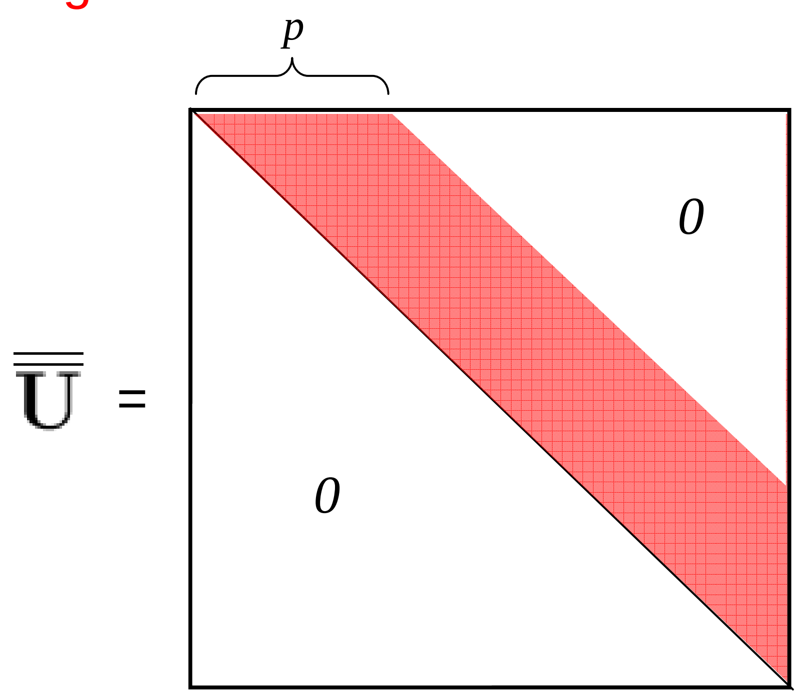
$$a_{ij} = a_{ji} \,, \quad |i-j| \le b$$
$$a_{ij} = a_{ji} = 0 \,, \quad |i-j| > b$$

b is half-bandwidth

Linear Systems of Equations
Special Matrices

Banded Coefficient Matrix
Gaussian Elimination
No Pivoting



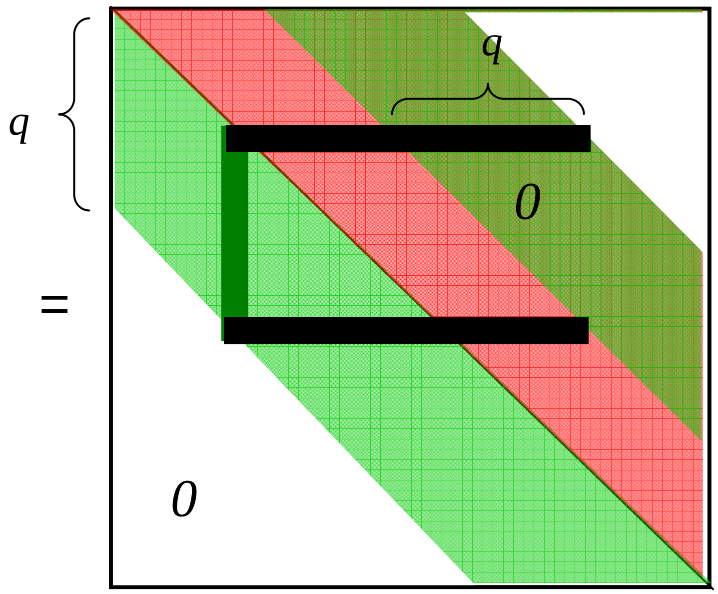


$$m_{ij} = 0, \; j > i, \; i > j + q$$

$$u_{ij} = 0, \; i > j, \; j > i + p$$

Linear Systems of Equations
Special Matrices

Banded Coefficient Matrix
Gaussian Elimination
With Pivoting

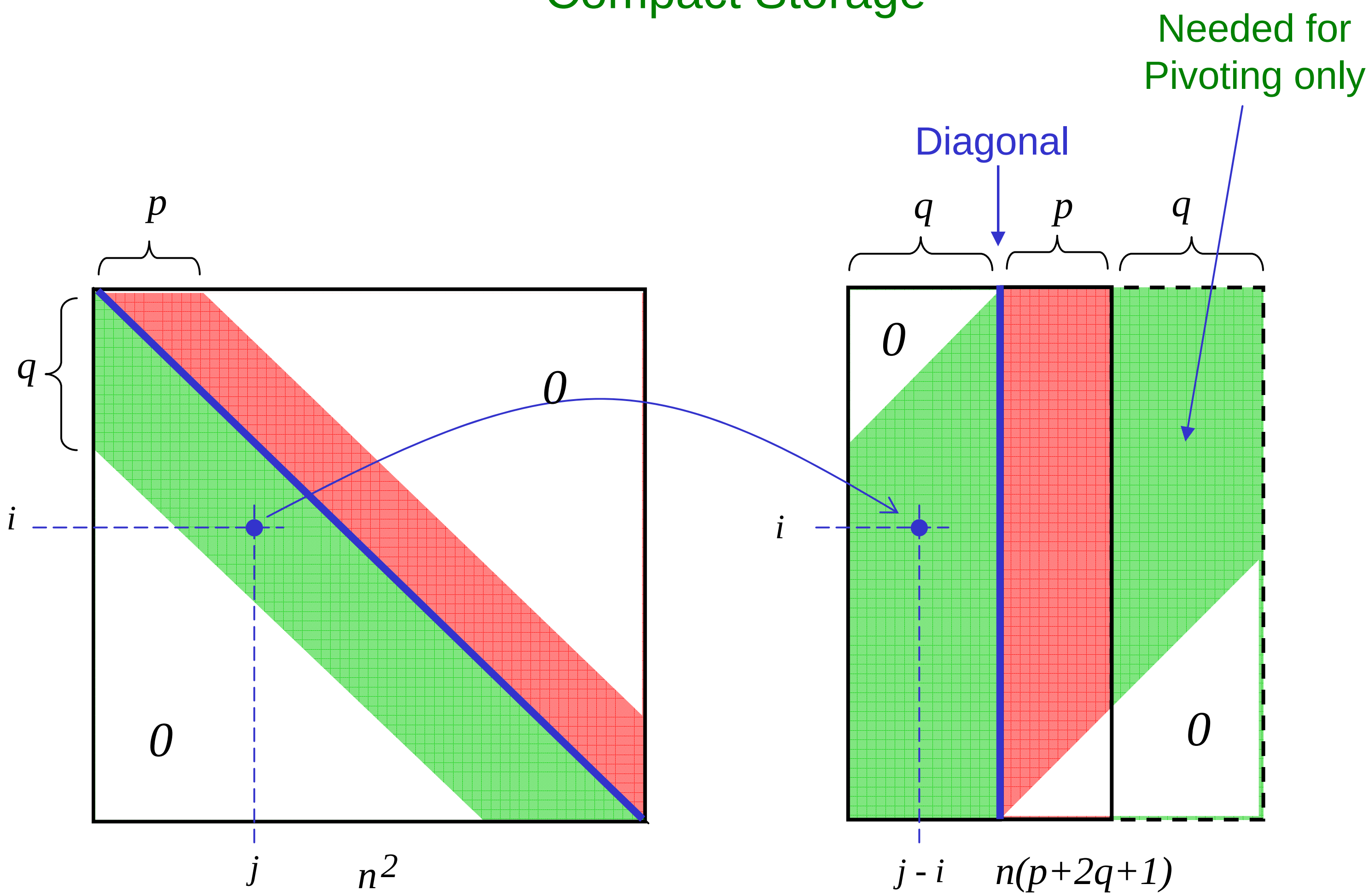


$$m_{ij} = 0,\ j > i,\ i > j + q$$

$$u_{ij} = 0,\ i > j,\ j > i + p + q$$

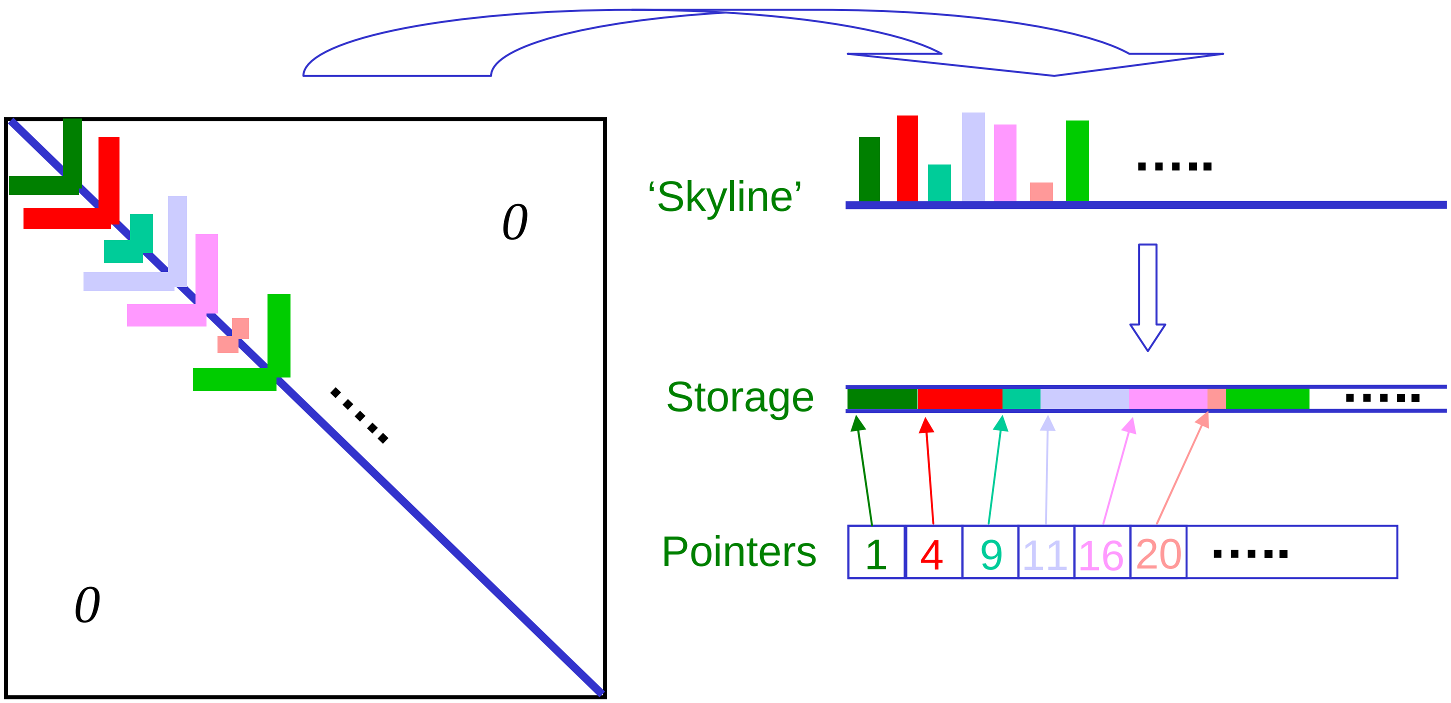
Linear Systems of Equations
Special Matrices

Banded Coefficient Matrix
Compact Storage



Linear Systems of Equations
Special Matrices

Sparse and Banded Coefficient Matrix
'Skyline' Systems



Skyline storage applicable when no pivoting is needed, e.g. for banded, symmetric, and positive definite matrices: FEM and FD methods. Skyline solvers are usually based on Choleski factorization

Linear Systems of Equations
Special Matrices

Symmetric, Positive Definite Coefficient Matrix

No pivoting needed

$$\overline{\overline{\mathbf{A}}} = \overline{\overline{\mathbf{L}}}\,\overline{\overline{\mathbf{U}}} = \overline{\overline{\mathbf{U}}}^{\dagger}\overline{\overline{\mathbf{U}}}$$

Choleski Factorization

$$\overline{\overline{\mathbf{U}}}^{\dagger} = [m_{ij}]$$

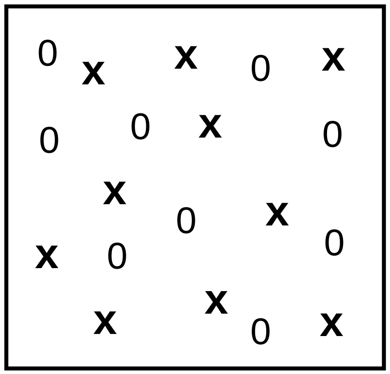
where

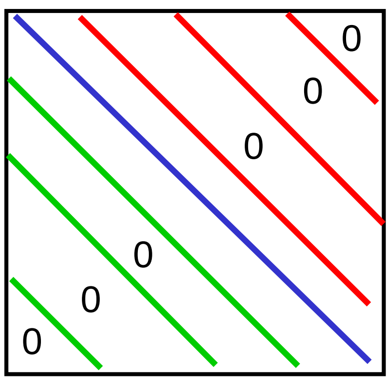
$$\left.\begin{aligned} m_{kk} &= \left(a_{kk} - \sum_{\ell=1}^{k-1} m_{k\ell}\overline{m}_{k\ell}\right)^{1/2} \\ m_{ik} &= \frac{a_{ik} - \sum_{\ell=1}^{k-1} m_{i\ell}\overline{m}_{k\ell}}{m_{kk}}, \; i = k+1, \ldots n \end{aligned}\right\} k = 1, \ldots n$$

$\dagger$ Complex Conjugate and Transpose

Linear Systems of Equations
Iterative Methods

Sparse, Full-bandwidth Systems





Rewrite Equations

$$\overline{\overline{\mathbf{A}}}\,\overline{\mathbf{x}} = \overline{\mathbf{b}} \quad \Leftrightarrow \quad \sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$a_{ii} \neq 0 \Rightarrow x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} \;,\; i = 1, \ldots n$$

Iterative, Recursive Methods

Jacobi's Method

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}} \;,\; i = 1, \ldots n$$

Gauss-Seidels's Method

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}} \;,\; i = 1, \ldots n$$

Linear Systems of Equations
Iterative Methods

Convergence

$$\left\|\overline{\mathbf{x}}^{(k+1)} - \overline{\mathbf{x}}\right\| \rightarrow 0 \text{ for } k \rightarrow \infty$$

Iteration – Matrix form

$$\overline{\mathbf{x}}^{(k+1)} = \overline{\overline{\mathbf{B}}}\overline{\mathbf{x}}^{(k)} + \overline{\mathbf{c}}\ , \ k = 0, \ldots$$

Decompose Coefficient Matrix

$$\overline{\overline{\mathbf{A}}} = \overline{\overline{\mathbf{D}}}\left(\overline{\overline{\mathbf{L}}} + \overline{\overline{\mathbf{I}}} + \overline{\overline{\mathbf{U}}}\right)$$

with

$$\overline{\overline{\mathbf{D}}} = \text{diag}\ a_{ii}$$

$$\overline{\overline{\mathbf{L}}} = \begin{cases} a_{ij}/a_{ii}\ , & i > j \\ 0, & i \leq j \end{cases}$$

Note: NOT LU-factorization

$$\overline{\overline{\mathbf{U}}} = \begin{cases} a_{ij}/a_{ii}\ , & i < j \\ 0, & i \geq j \end{cases}$$

Jacobi's Method

$$\overline{\mathbf{x}}^{(k+1)} = -\left(\overline{\overline{\mathbf{L}}} + \overline{\overline{\mathbf{U}}}\right)\overline{\mathbf{x}}^{(k)} + \overline{\overline{\mathbf{D}}}^{-1}\overline{\mathbf{b}}$$

Iteration Matrix form

$$\overline{\overline{\mathbf{B}}} = -\left(\overline{\overline{\mathbf{L}}} + \overline{\overline{\mathbf{U}}}\right)$$

$$\overline{\mathbf{c}} = \overline{\overline{\mathbf{D}}}^{-1}\overline{\mathbf{b}}$$

Convergence Analysis

$$\begin{aligned} \overline{\mathbf{x}}^{(k+1)} &= \overline{\overline{\mathbf{B}}}\overline{\mathbf{x}}^{(k)} + \overline{\mathbf{c}} \\ \overline{\mathbf{x}} &= \overline{\overline{\mathbf{B}}}\overline{\mathbf{x}} + \overline{\mathbf{c}} \\ \hline \overline{\mathbf{x}}^{(k+1)} - \overline{\mathbf{x}} &= \overline{\overline{\mathbf{B}}}\left(\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right) \\ &= \overline{\overline{\mathbf{B}}} \cdot \overline{\overline{\mathbf{B}}}\left(\overline{\mathbf{x}}^{(k-1)} - \overline{\mathbf{x}}\right) \\ &\cdot \\ &= \overline{\overline{\mathbf{B}}}^{k+1}\left(\overline{\mathbf{x}}^{(0)} - \overline{\mathbf{x}}\right) \end{aligned}$$

$$\left\|\overline{\mathbf{x}}^{(k+1)} - \overline{\mathbf{x}}\right\| \leq \left\|\overline{\overline{\mathbf{B}}}^{k+1}\right\| \left\|\overline{\mathbf{x}}^{(0)} - \overline{\mathbf{x}}\right\| \leq \left\|\overline{\overline{\mathbf{B}}}\right\|^{k+1} \left\|\overline{\mathbf{x}}^{(0)} - \overline{\mathbf{x}}\right\|$$

Sufficient Convergence Condition

$$\left\|\overline{\overline{\mathbf{B}}}\right\| < 1$$

Linear Systems of Equations
Iterative Methods

Sufficient Convergence Condition

$$\left\|\overline{\overline{\mathbf{B}}}\right\| < 1$$

Jacobi's Method

$$b_{ij} = -\frac{a_{ij}}{a_{ii}}\ ,\ i \neq j$$

$$\left\|\overline{\overline{\mathbf{B}}}\right\|_\infty = \max_i \sum_{j=1,j\neq i}^{n} \frac{|a_{ij}|}{|a_{ii}|}$$

Sufficient Convergence Condition

$$\sum_{j=1,j\neq i}^{n} |a_{ij}| < |a_{ii}|$$

Diagonal Dominance

Stop Criterion for Iteration

$$\begin{aligned}\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}} &= \overline{\overline{\mathbf{B}}}\left(\overline{\mathbf{x}}^{(k-1)} - \overline{\mathbf{x}}\right) \qquad \pm\overline{\overline{\mathbf{B}}}\,\overline{\mathbf{x}}^{(k)} \\ &= -\overline{\overline{\mathbf{B}}}\left(\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}^{(k-1)}\right) + \overline{\overline{\mathbf{B}}}\left(\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right)\end{aligned}$$

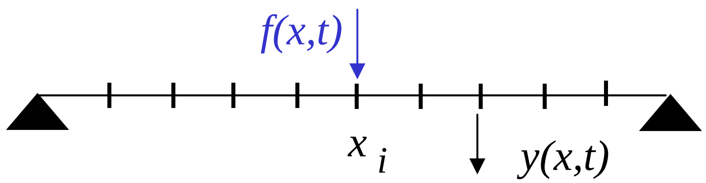
$$\left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right\| \leq \left\|\overline{\overline{\mathbf{B}}}\right\| \left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}^{(k-1)}\right\| + \left\|\overline{\overline{\mathbf{B}}}\right\| \left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right\|$$

$$\left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right\| \leq \frac{\left\|\overline{\overline{\mathbf{B}}}\right\|}{1 - \left\|\overline{\overline{\mathbf{B}}}\right\|} \left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}^{(k-1)}\right\|$$

$$\left\|\overline{\overline{\mathbf{B}}}\right\| < 1/2 \Rightarrow \left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right\| \leq \left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}^{(k-1)}\right\|$$

Linear Systems of Equations
Tri-diagonal Systems

Forced Vibration of a String



Harmonic excitation

$$f(x,t) = f(x)\cos(\omega t)$$

Differential Equation

$$\frac{d^2y}{dx^2} + k^2 y = f(x)$$

Boundary Conditions

$$y(0) = 0\ , \quad y(L) = 0$$

Finite Difference

$$\left.\frac{d^2y}{dx^2}\right|_{x_i} \simeq \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

Discrete Difference Equations

$$y_{i-1} + \left((kh)^2 - 2\right) y_i - y_{i+1} = f(x_i)h^2$$

Matrix Form

$$\begin{bmatrix} (kh)^2-2 & 1 & . & . & . & . & 0 \\ 1 & (kh)^2-2 & 1 & & & & . \\ . & & . & . & & & . \\ . & & 1 & (kh)^2-2 & 1 & & . \\ . & & & . & . & & . \\ 0 & . & . & . & . & 1 & (kh)^2-2 \end{bmatrix} \overline{\mathbf{x}} = \begin{Bmatrix} f(x_1)h^2 \\ . \\ . \\ f(x_i)h^2 \\ . \\ . \\ f(x_n)h^2 \end{Bmatrix}$$

Tridiagonal Matrix

$kh < 1$ Symmetric, positive definite: No pivoting needed

vib_string.m

```
n=99;
L=1.0;
h=L/(n+1);
k=2*pi;
kh=k*h
x=[h:h:L-h]';
a=zeros(n,n);
f=zeros(n,1);
o=1
```

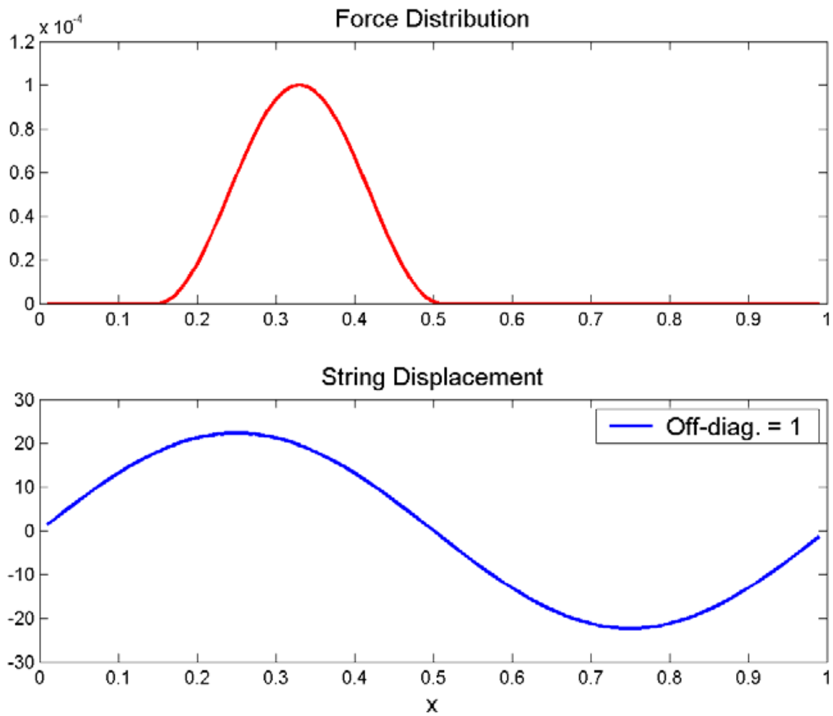
Off-diagonal values

```
a(1,1) =kh^2 - 2;
a(1,2)=o;

for i=2:n-1
    a(i,i)=a(1,1);
    a(i,i-1) = o;
    a(i,i+1) = o;
end
a(n,n)=a(1,1);
a(n,n-1)=o;
% Hanning windowed load
nf=round((n+1)/3);
nw=round((n+1)/6);
nw=min(min(nw,nf-1),n-nf);
nw1=nf-nw;
nw2=nf+nw;
f(nw1:nw2) = h^2*hanning(nw2-nw1+1);

figure(1)
hold off
subplot(2,1,1); plot(x,f,'r');
% Exact solution
y=inv(a)*f;
subplot(2,1,2); plot(x,y,'b');
```

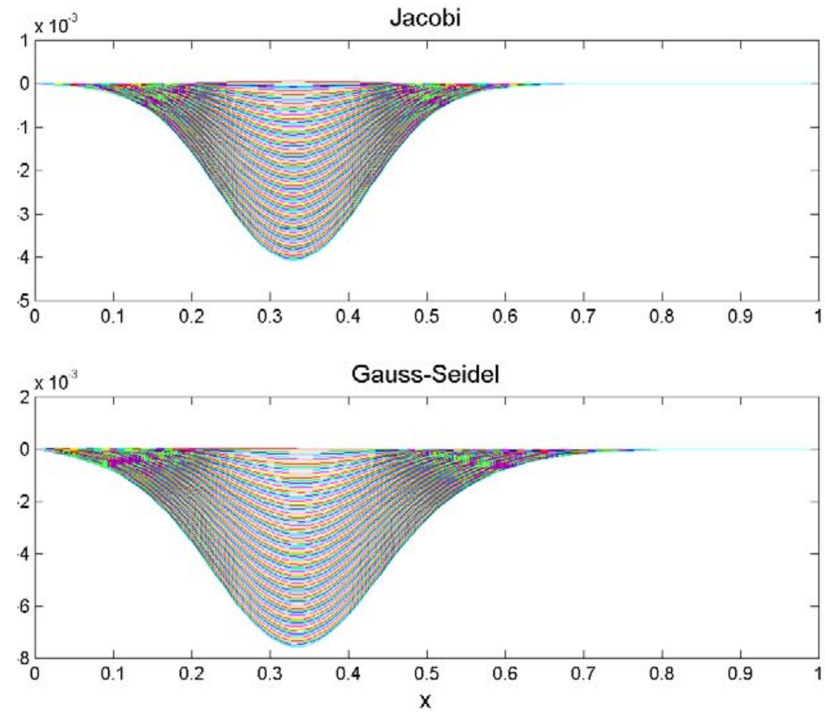
```
% Iterative solution using Jacobi and Gauss-Seidel
b=-a;
c=zeros(n,1);
for i=1:n
    b(i,i)=0;
    for j=1:n
        b(i,j)=b(i,j)/a(i,i);
        c(i)=f(i)/a(i,i);
    end
    end

nj=100;
xj=f;
xgs=f;

figure(2)
nc=6
col=['r' 'g' 'b' 'c' 'm' 'y']
hold off
for j=1:nj
    xj=b*xj+c;
    xgs(1)=b(1,2:n)*xgs(2:n) + c(1);
    for i=2:n-1
        xgs(i)=b(i,1:i-1)*xgs(1:i-1) + b(i,i+1:n)*xgs(i+1:n) +c(i);
    end
    xgs(n)= b(n,1:n-1)*xgs(1:n-1) +c(n);
    cc=col(mod(j-1,nc)+1);
    subplot(2,1,1); plot(x,xj,cc); hold on;
    subplot(2,1,2); plot(x,xgs,cc); hold on;
    hold on
end
```

vib_string.m
o=1.0

Exact Solution



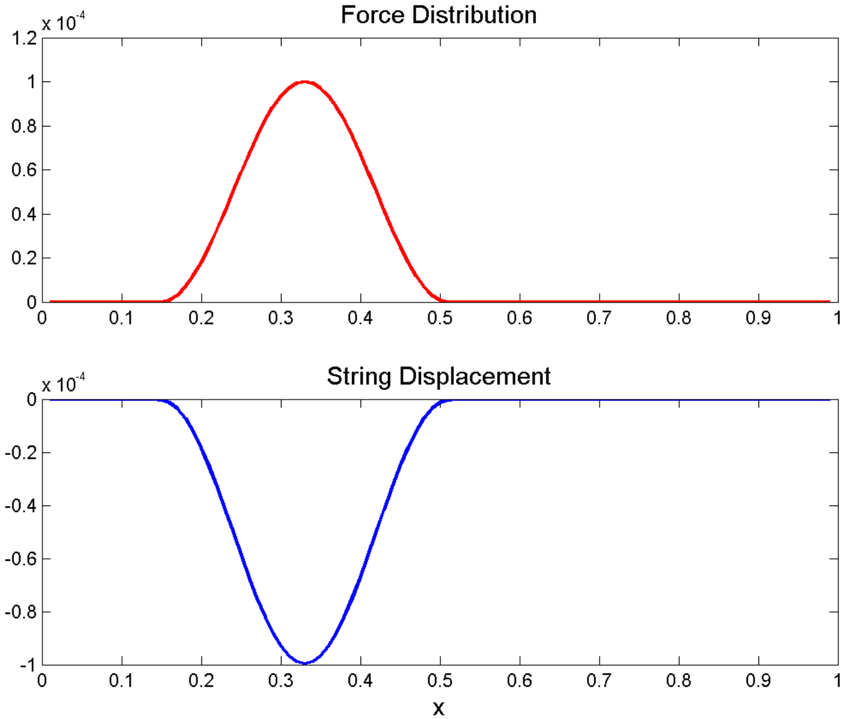
Iterative Solutions



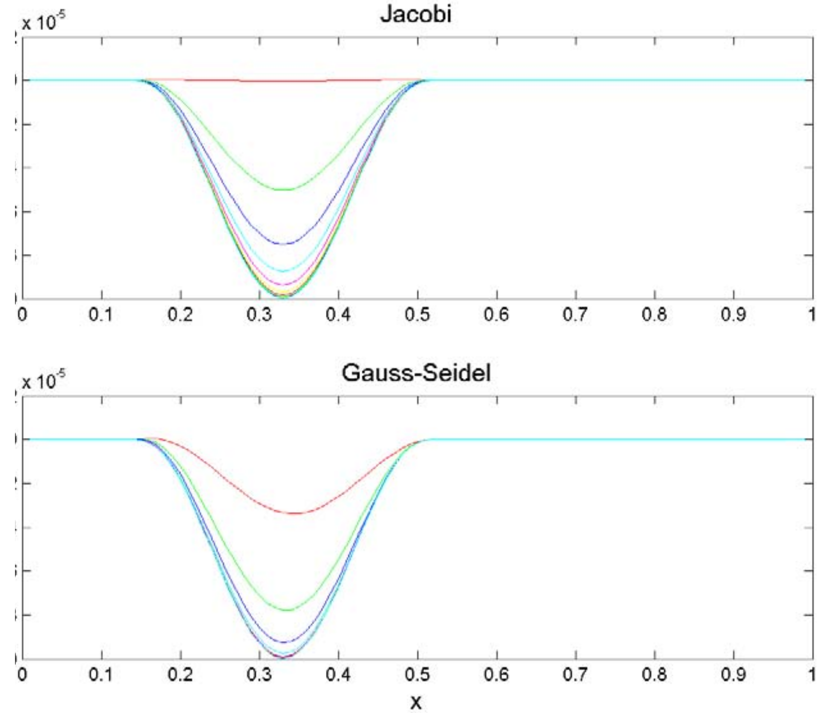
Coefficient Matrix Not Strictly Diagonally Dominant

vib_string.m
o = 0.5

Exact Solution



Iterative Solutions



Coefficient Matrix Strictly Diagonally Dominant