Root Finding

Suppose you wish to find the wave number, k, of gravity water waves with a frequency, f, of 0.2 Hz. in water that is 5 meters deep. The circular frequency of 0.2 Hz. waves is $\omega = 2\pi f = 1.2566$ radians/second. The dispersion relation for gravity water waves is:

$$kg \tanh kh = \omega^2$$

g is the acceleration of gravity, 9.81 m/s^2 and h is the water depth, 5 m.

This equation can be written as:

$$kg \tanh kh - \omega^2 = 0$$

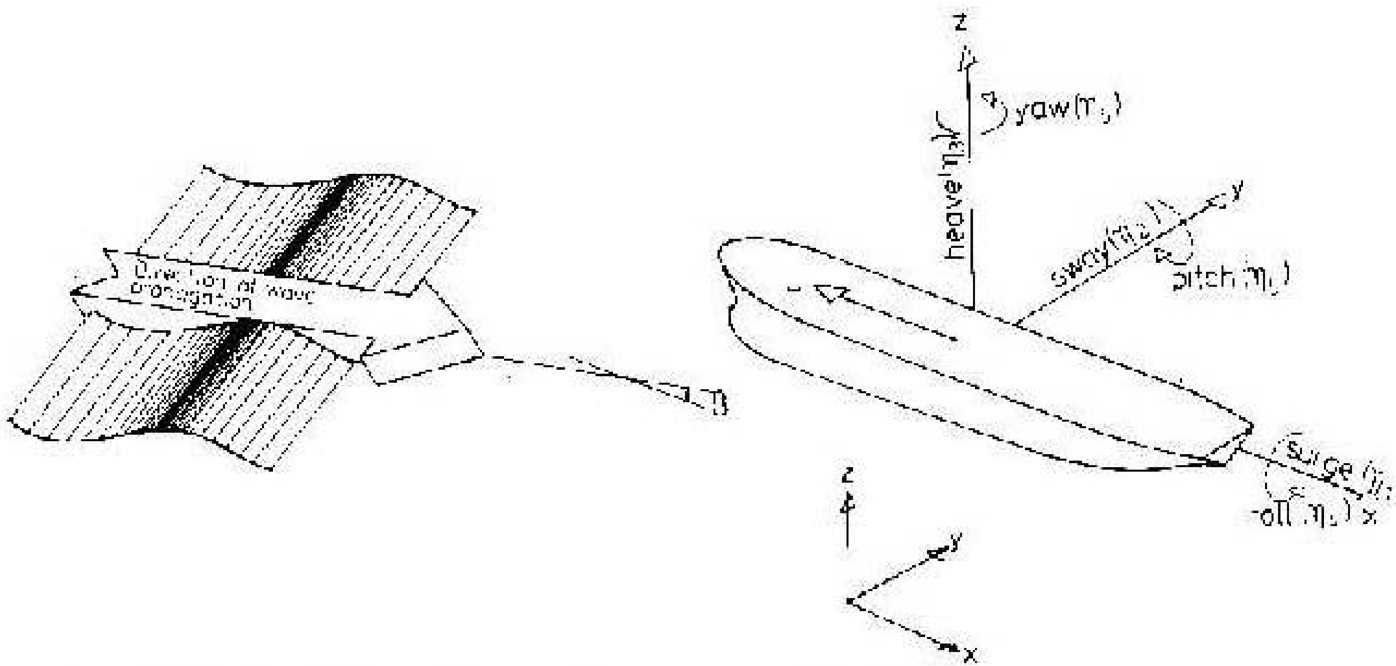
If we write an equation: $y(k) = kg \tanh kh - \omega^2$,

The problem at hand is the same as asking: "What is the value of k such that $y(k) = 0$? The value of a quantity that makes another equal to zero is called a root and the question above is called *Root Finding.*

```
% biseck program to find k given omega
% using the bisection root finding method
om = 1.2566;
g = 9.81;
h = 5.0;
k1 = 0.0;
k2 = 0.5;
k3 = 0.25;
y = k3*g*tanh(k3*h) - om^2;
for m = 1:50
    y = k3*g*tanh(k3*h) - om^2;
   if (y*y < 1.0e-8);
        break
    end
    if(y >= 0.0);
        k2 = k3;
        k3 = 0.5*(k1+k2);
    elseif (y <= 0.0) ;
        k1 = k3;
        k3 = 0.5*(k1  + k2);
    else;
        fprintf(1,'there was no root')
    end;
end ;
fprintf(1,' k = %8.4f\n',k3);
fprintf(1,' Number of iterations =%3.0f\n',m)
```

```
>> biseck
 k =   0.2073
 Number of iterations = 15
>>
```

Six-Degree-of-Freedom Motion of a Ship in Waves



Translation in x: surge $\eta_1(t)$;
Translation in y: sway $\eta_2(t)$;
Translation in z: heave $\eta_3(t)$;
Rotation with x: roll $\eta_4(t)$;
Rotation with y: pitch $\eta_5(t)$;
Rotation with z: yaw $\eta_6(t)$;

Solution of Equation of Motion of a Ship in Waves

Equation of Motion:

$$\sum_{k=1}^{6}\left[(M_{jk}+A_{jk})\frac{d^2}{dt^2}\eta_k + B_{jk}\frac{d}{dt}\eta_k + C_{jk}\eta_k\right] = F_j(t) \qquad (j=1,\ldots,6)$$

In a state-sate, ship has a periodic response: $\eta_k(t) = \zeta_k e^{i\omega t}, k = 1, \ldots, 6.$

the wave excitation: $F_j(t) = f_j e^{i\omega t}, j = 1, \ldots, 6.$

The equation of motion becomes:

$$\sum_{k=1}^{6}\left[-\omega^2(M_{jk}+A_{jk})\zeta_k + i\omega B_{jk}\zeta_k + C_{jk}\zeta_k\right] = f_j \qquad (j=1,\ldots,6)$$

In a matrix form, it becomes: $\{-\omega^2([M]+[A]) + i\omega[B] + [C]\}\{\zeta\} = \{f\}$

Thus, we have $\{\zeta\} = [E]^{-1}\{f\}$ $\qquad [E] = -\omega^2([M]+[A]) + i\omega[B] + [C]$

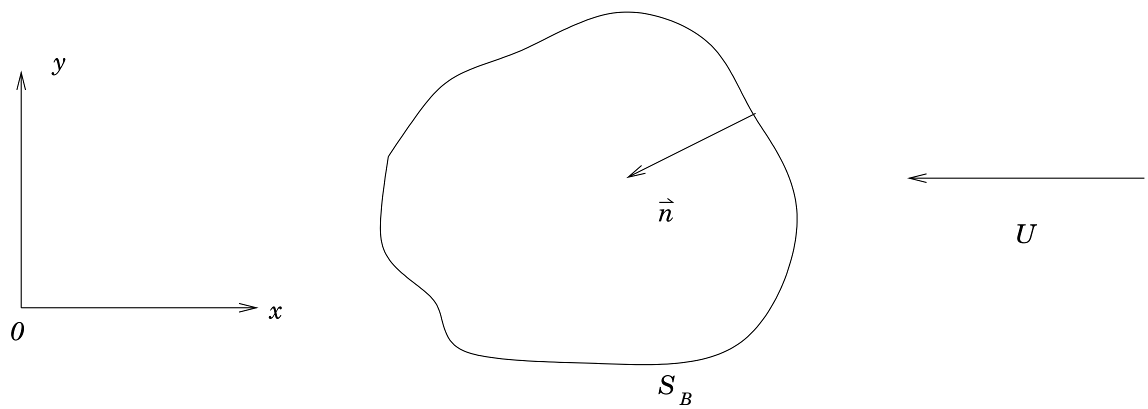
The key is to determine the 6×6 matrices: added mass [A], damping [B], restoring coefficients [C], and 6×1 vector: excitation {f}

where

2.29 - Numerical Fluid Mechanics	Spring 2007

Lecture 9 - Uniform Flow Past an Arbitrary Body

Dr.Yuming Liu



$$\Phi(x,y) = -Ux + \varphi(x,y)$$

On S_B: $\Phi_n = 0 \quad \longrightarrow \quad \varphi_n = Un_x$

In the fluid: $\nabla^2\Phi = 0 \quad \longrightarrow \quad \nabla^2\varphi = 0$

In the far field: $\nabla\varphi = 0$

- Purpose: To find $\varphi(x,y)$ on the body surface (S_B). After knowing φ on S_B, flow velocity and pressure on S_B can be determined easily.

Apply Green's Theorem $\longrightarrow$ a boundary integral equation:

$$\int_{S_B} \varphi(\xi,\eta)\frac{\partial G}{\partial n} d_{\ell_{\xi\eta}} + \pi\varphi(x,y) = \int_{S_B} \varphi_n(\xi,\eta) G d_{\ell_{\xi\eta}}$$
$$G(x,y;\xi,\eta) = -\ln\sqrt{(x-\xi)^2+(y-\eta)^2}$$

To solve the above integral equation, we use the panel method.

- Geometry approximation: divide S_B into N segments, approximate each segment by a straight line segment.
- Over each segment, approximate the unknown potential by a constant.

$$\Longrightarrow \boxed{\pi\varphi(x,y) + \sum_{j=1}^{N}\varphi_j \int_{S_j} \frac{\partial G}{\partial n} d_\ell = \sum_{j=1}^{N} \varphi_{nj} \int_{S_j} G d_\ell}^{(*)}$$

Applying equation ($*$) at i^{th} panel, i.e. letting $x = x_i$ and $y = y_i$, we have

$$\longrightarrow \pi\varphi_i + \sum_{j=1}^{N} \varphi_j \underbrace{\int_{S_j} \frac{\partial G(x_i, y_i)}{\partial n_j} d_\ell}_{P_{ji}} = \underbrace{\sum_{j=1}^{N} \varphi_{nj} \int_{S_j} G(x_i, y_i) d_\ell}_{b_i}$$

$$\underbrace{[\pi\delta_{ij} + P_{ji}]}_{A}\{\varphi_i\} = \{b_i\}, \qquad i = 1, \cdots, N$$

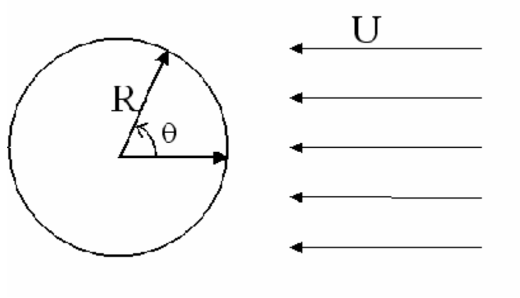
$$j = 1, \cdots, N$$

$$\longrightarrow \quad [A]\{\varphi\} = \{b\}$$

$[A] : N \times N, \{\varphi\}, \{b\} : 1 \times N$

$[A]$: Diagonal dominant full matrix.

Steady Uniform Flow Past a Stationary Object



$$\Phi = -Ux + \phi$$

We first consider the case of a circular cylinder. For the following numerical computations, set the radius of the cylinder $R = 1$ and the speed of free stream U=1.

(1) Disturbance potential ϕ. By applying the no-flux boundary condition on the cylinder surface, calculate the *disturbance* velocity potential ϕ on the cylinder surface. Plot your numerical result of ϕ and the analytical result as a function of θ for the number of panels N=40.

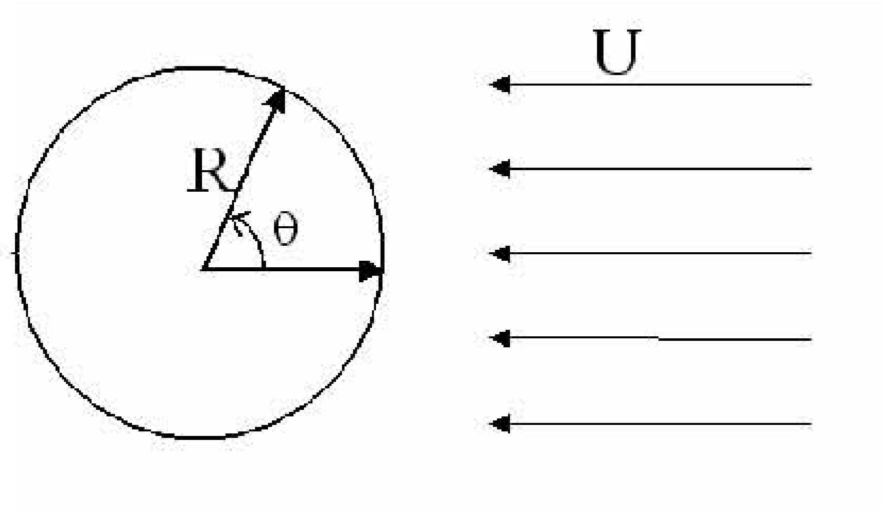
(2) Convergence of the numerical method. Compute the maximum and average errors of the numerical solution of the disturbance potential ϕ for N=10, 20, 40, 80, and 160. The numerical error here is defined to be the absolute value of the difference between the numerical result and analytical solution. Plot the maximum and averaged errors vs. N to determine the convergence rate of the Constant Panel Method for this problem. (*loglog* plot is recommended for this question).

(3) Tangential velocity. Use a finite difference formula to compute the tangential velocity of the *total* flow on the body surface. Plot your numerical result and the analytical result as a function of θ for the number of panels N=40.

(4) Hydrodynamic force. Numerically integrate the hydrodynamic pressure on the body to compute the horizontal and vertical hydrodynamic forces on the body. Compare your numerical solutions (with N=40) to the analytical solution.

(5) Added mass coefficients. Calculate the added mass coefficients m_{xx} and m_{yx} of the circular cylinder using the formula: $m_{xx} = \dfrac{\rho \int_S \phi n_x dl}{\rho \pi R^2}$ and $m_{yx} = \dfrac{\rho \int_S \phi n_y dl}{\rho \pi R^2}$. Compare your numerical results (with N=20, 40, and 80) to the analytical solution: $m_{xx} = 1, m_{yx} = 0$. Note that ϕ here is the disturbance potential obtained in (1).

Convergence of the Solution with Number of Panels



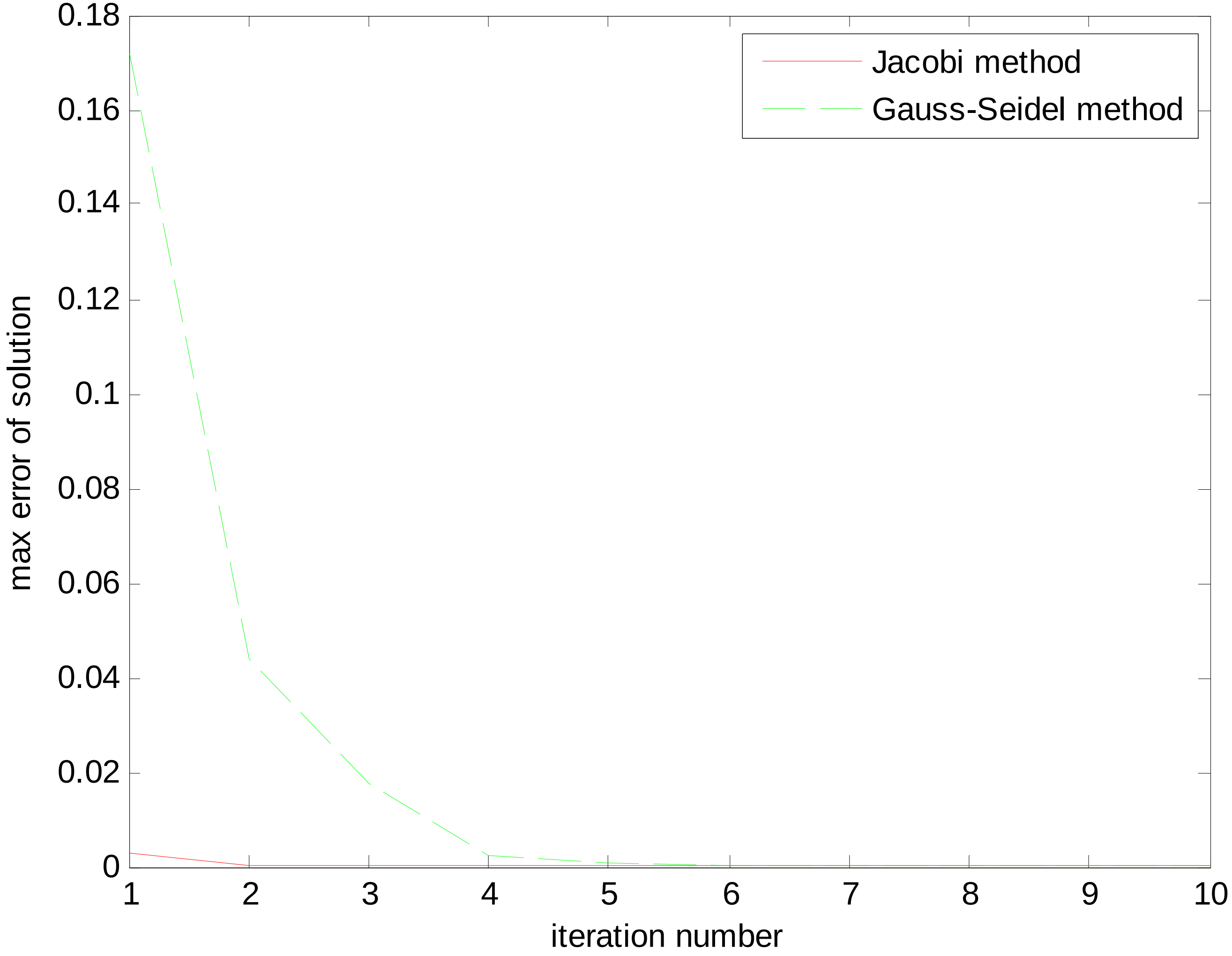
$$\Phi = -Ux + \phi$$

Gaussian elimination or LU Factorization:

# of panels	Max error in velocity potential	Error in added mass m_{xx}
N=20	0.0161	0.0040
N=40	0.0041	0.0010
N=80	0.0010	0.0003
N=160		0.0001

0.00026

Convergence of iteration: Jacobi vs. Gauss-Seidel

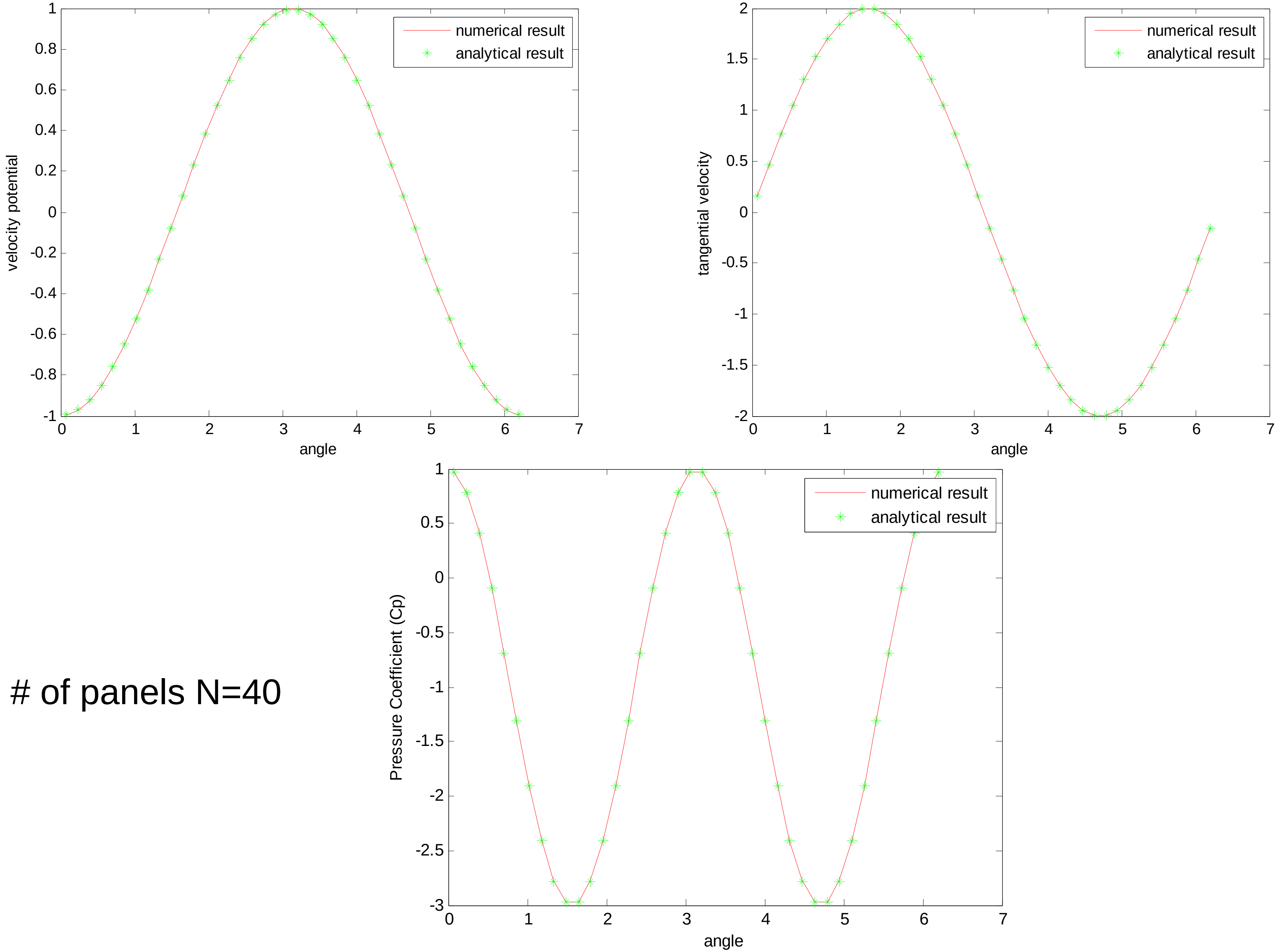


Max Error: Compared to the solution by Gaussian elimination
# of panels N=40

Jacobi iteration:
1 : 3.077659e-003
2 : 9.491704e-006
3 : 2.927366e-008
4 : 9.030277e-011
5 : 2.793321e-013
6 : 1.332268e-015
7 : 8.881784e-016
8 : 8.881784e-016
9 : 8.881784e-016
10 : 8.881784e-016

Gauss-Seidel:
1 : 1.720594e-001
2 : 4.416175e-002
3 : 1.778993e-002
4 : 2.141429e-003
5 : 7.727595e-004
6 : 1.183172e-004
7 : 3.416042e-005
8 : 6.943812e-006
9 : 1.505766e-006
10 : 3.756878e-007

Comparison of Numerical Solution with Analytic Solution



# of panels N=40