18.435/2.111 Homework # 5 Solutions

Solution to 1: We want

$$\frac{1}{3}\left(|\,0\rangle\langle 0\,| + \frac{1}{4}(|\,0\rangle + \sqrt{3}\,|\,1\rangle)(\langle 0\,| + \sqrt{3}\,\langle 1\,|) + \frac{1}{4}(|\,0\rangle - \sqrt{3}\,|\,1\rangle)(\langle 0\,| - \sqrt{3}\,\langle 1\,|)\right)$$

which is

$$\frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{12}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix} + \frac{1}{12}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Solution to 2: When we take the partial trace over the second qubit of the state

$$\frac{1}{\sqrt{3}}\left(|\,00\rangle + |\,01\rangle + |\,10\rangle\right),$$

we can compute the density matrix of the above state

$$\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and taking the partial trace explicitly, we obtain

$$\frac{1}{3}\begin{pmatrix} \mathrm{Tr}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ \mathrm{Tr}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} & \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Solution to 3:
After we apply the controlled σ_z to

$$|\,\psi\rangle \otimes \left(\frac{\sqrt{3}}{2}\,|\,0\rangle + \frac{1}{2}\,|\,1\rangle\right)$$

we get the state

$$\frac{\sqrt{3}}{2}\,|\,\psi\rangle \otimes |\,0\rangle + \frac{1}{2}\sigma_z\,|\,\psi\rangle \otimes |\,1\rangle\,.$$

Now, we can take the partial trace by measuring the second qubit in the $|\,0\rangle\,, |\,1\rangle$ basis and using the resulting states of the first qubit and their probabilities to compute the density matrix of the second qubit. If we do this with the above state, we get

$$\frac{3}{4}\,|\,\psi\rangle\langle\psi\,| + \frac{1}{4}\sigma_z\,|\,\psi\rangle\langle\psi\,|\,\sigma_z^\dagger$$

which is easy to see how to write in operator sum notation. We get

$$A_1 = \frac{\sqrt{3}}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using a different measurement on the second qubit gives alternative operator sum decompositions.

Solution to 4:

We want to compose two noisy operations. The first one takes

$$\rho \to \sum_i B_i \rho B_i^\dagger$$

where

$$B_1 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_2 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and the second one takes

$$\rho \to \sum_i A_i \rho A_i^\dagger$$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & \sqrt{q} \\ 0 & 0 \end{pmatrix}.$$

Putting them together, one sees the four operations in the operator sum notation are A_1B_1, A_1B_2, A_2B_1, and A_2B_2. However,

$$A_2B_1 = \sqrt{1-p}A_2 \quad \text{and} \quad A_2B_2 = -\sqrt{p}A_2$$

These can be combined into one operation, since

$$\begin{aligned} A_2B_1\rho B_1^\dagger A_2^\dagger + A_2B_2\rho B_2^\dagger A_2^\dagger &= (1-p)A_2\rho A_2^\dagger + pA_2\rho A_2^\dagger \\ &= A_2\rho A_2^\dagger. \end{aligned}$$

Thus, we get a noisy quantum operation with an operator-sum expression having just three operators:

$$A_1B_1 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix} \qquad A_1B_2 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{1-q} \end{pmatrix}. \qquad A_2 = \begin{pmatrix} 0 & \sqrt{q} \\ 0 & 0 \end{pmatrix}.$$

Solution to 5:
We can rewrite the depolarizing operation $\mathcal{D}$ as

$$\mathcal{D}(\rho) = (1 - \frac{4p}{3})\rho + \frac{4p}{3}\frac{I}{2}$$

Using this formulation, it is clear that if the eigenvalues of ρ are a and b, the eigenvalues of $\mathcal{D}(\rho)$ are $(1-4p/3)a + 2p/3$ and $(1-4p/3)b + 2p/3$. (If it's not clear, consider that when you change the basis to diagonalize ρ, the above formulation is unchanged.) Since $a, b \geq 0$, the eigenvalues of $\mathcal{D}(\rho)$ are larger than $2p/3$.

Solution to 6:
Reformulating the problem, we want to find the relation between

$$|\, x + C_2\rangle = \sum_{y \in C_2} |\, x + y\rangle$$

and

$$|\, x + C_2\rangle_{u,v} = \sum_{y \in C_2} (-1)^{u \cdot y} \,|\, x + y + v\rangle \,.$$

Suppose we take the first code $|\, x + C_2\rangle$ and first apply a σ_z to all the qubits that are 1's in u, and then a σ_x to the position of all the 1's in v. we get

$$|\, x + C_2\rangle_\alpha = \sum_{y \in C_2} (-1)^{u\cdot(x+y)} \,|\, x + y + v\rangle \,.$$

This is

$$(-1)^{u \cdot x} \,|\, x + C_2\rangle_{u,v} \,.$$

Thus, the second CSS code (with u, v) can be obtained by first applying a unitary transformation U_u to the state being encoded, then encoding it using the first CSS code, and finally applying σ_z to some encoding qubits and σ_x to other encoding qubits. This unitary transformation U_u is

$$|\, x + C_2\rangle \to (-1)^{u \cdot x} \,|\, x + C_2\rangle \,.$$

Applying a unitary transformation to the encoded state doesn't affect the error correcting properties of the code, since the code is supposed to protect all allowed codewords. Applying the Pauli matrices σ_x and σ_z to specific qubits in the code also doesn't affect the overall error correcting properties of the code, since up to a possible overall -1 sign in the global phase, this operation takes phase errors to phase errors and bit errors to bit errors.