

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering
2.004 Dynamics and Control II
Fall 2007

Problem Set #7

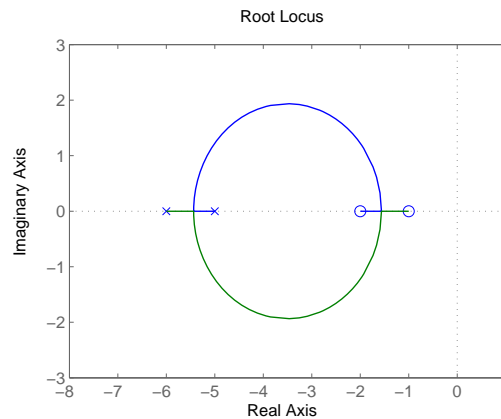
Solution

Posted: Friday, Nov. 2, '07

1. Nise problem 5 from chapter 8, page 476.

Answer: The open loop transfer function is

$$G(s) = \frac{K(s+1)(s+2)}{(s+5)(s+6)}.$$



To find the break-in and breakaway points, we apply rule 6 as follows:

$$K(\sigma) = -\frac{(\sigma+5)(\sigma+6)}{(\sigma+1)(\sigma+2)} = -\frac{\sigma^2 + 11\sigma + 30}{\sigma^2 + 3\sigma + 2}.$$

Taking the derivative,

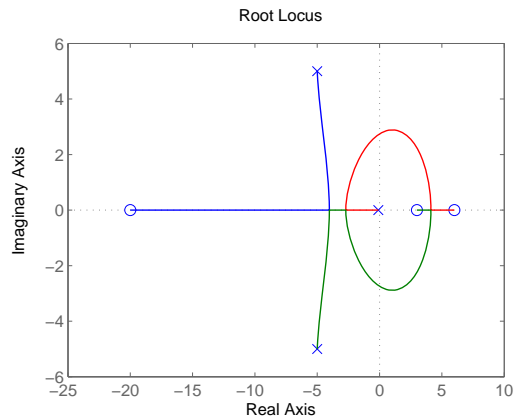
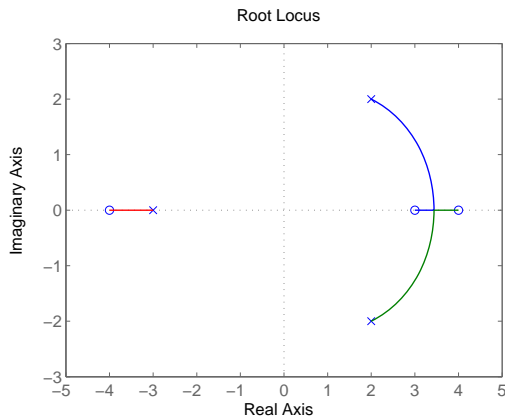
$$\frac{dK}{d\sigma} = -\frac{-8\sigma^2 - 56\sigma - 68}{(\sigma^2 + 3\sigma + 2)^2} = -4\frac{-2\sigma^2 - 14\sigma - 17}{(\sigma^2 + 3\sigma + 2)^2}.$$

and setting $dK/d\sigma = 0$, we find $\sigma_1 = -1.5635$ and $\sigma_2 = -5.4365$.

2. Nise problem 7 from chapter 8, page 477.

Answer: From rule 3 about the real-axis segment, we know that the root locus should exist between the two zeros in the right-hand plane as well as the pole and zero in the left-hand plane. Next step is to deal with the two poles with

imaginary parts. The root locus should depart from the two poles. It cannot go to the infinity because we already have the same number of poles and zeros. Therefore it has to be joined one of the two real-axis segment. The root locus departing from the two poles eventually should arrive at the two zeros in the right-hand plane. Here two possibilities arise: 1) The root locus directly is joined to the real-axis segment between the two zeros in right-hand plane, or 2) It first cuts through the real-axis segment between the pole and zero in the left-hand plane, and comes back to the right-hand plane to join the real-axis segment between the two zeros. Which option turns out to be the actual root locus depends on the relative location of the open-loop poles and zeros. The two plots below show two examples obtained using MATLAB .



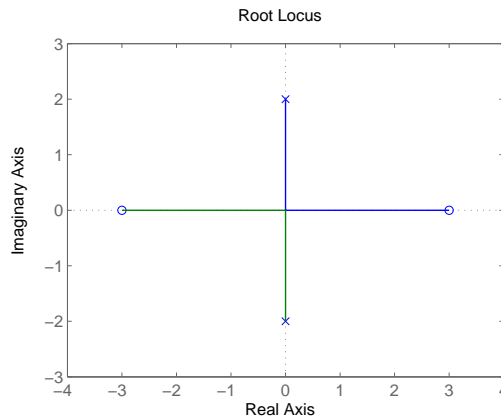
3. Nise problem 9 from chapter 8, page 477.

Answer:

The open-loop transfer function is

$$G(s) = \frac{K(s^2 - 9)}{(s^2 + 4)} = \frac{K(s - 3)(s + 3)}{(s - 2j)(s + 2j)},$$

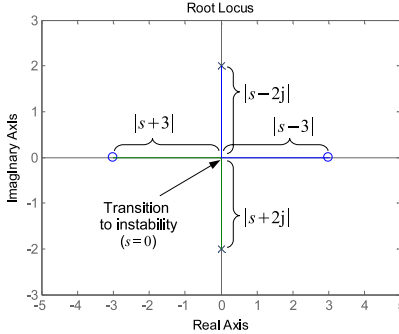
and the root locus is shown below;



Clearly, the system becomes unstable for gain equal to or larger than the gain required to bring the closed-loop poles together to a double pole at the origin. This critical gain is given by

$$\begin{aligned} K &= \frac{1}{|G(s)|_{s=0}} \\ &= \left. \frac{s^2 + 4}{s^2 - 9} \right|_{s=0} = \frac{|s + 2j||s - 2j|}{|s + 3||s - 3|} \bigg|_{s=0} = \frac{2 \times 2}{3 \times 3} = \frac{4}{9}. \end{aligned}$$

See the geometrical interpretation below.



Alternatively, we can find K from the relationship $KG(s) + 1 = 0$, which gives

$$-s^2 - 4 = K(s^2 - 9),$$

thus

$$s^2 = \frac{9K - 4}{K + 1}.$$

If $\frac{9K-4}{K+1} > 0$ (or $K > 4/9$), then we get two real poles at $s = \pm\sqrt{\frac{9K-4}{K+1}}$, which make the system unstable because of the pole in the right-hand plane. If $\frac{9K-4}{K+1} < 0$ (or $K < 4/9$), then we obtain two pure imaginary poles at $s = \pm j\sqrt{\frac{4-9K}{K+1}}$, which make the system marginally stable. If $K = 4/9$, then two double poles are at $s = 0$.

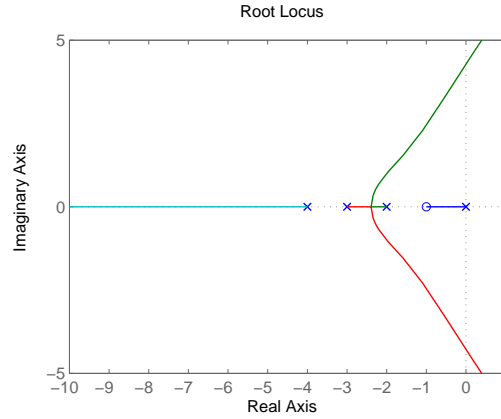
4. Nise problem 17 from chapter 8, page 479.

Answer:

The open-loop transfer function is

$$G(s) = \frac{K(s + 1)}{s(s + 2)(s + 3)(s + 4)}.$$

- a. Sketch the root locus.



b. The asymptotes.

By rule 5,

$$\sigma_a = \frac{(-4 - 3 - 2 - 0) - (-1)}{4 - 1} = -\frac{8}{3},$$

$$\theta_a = \frac{(2m + 1)\pi}{4 - 1} = \left\{ \frac{\pi}{3}, \pi, \frac{5\pi}{3} \right\}.$$

c. The value of gain that makes the system marginally stable.

By rule 7, setting $G(j\omega) = -1$, we find

$$\frac{K(j\omega + 1)}{(j\omega)^4 + 9(j\omega)^3 + 26(j\omega)^2 + 24(j\omega)} = -1.$$

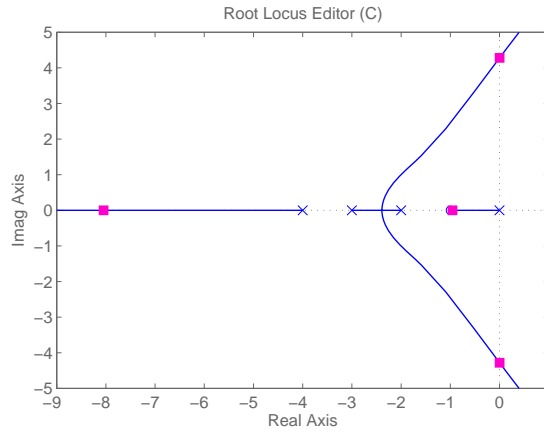
Rearranging,

$$K(j\omega + 1) = -(\omega^4 - j9\omega^3 - 26\omega^2 + 24j\omega),$$

we end up with two equations for the real and imaginary parts, respectively:

$$\begin{aligned} \omega^4 - 26\omega^2 + K &= 0, \\ 9\omega^3 - 24\omega - K\omega &= 0. \end{aligned}$$

From the second equation, $\omega^2 = (24 + K)/9$. Inserting this result to the first equation, we find $K = 140.7964$. (The other value is $K = -35.7964$, which is invalid because it is negative.)



Indeed, at gain $K = 140.7964$, the root locus crosses the imaginary axis in the MATLAB simulation as well.

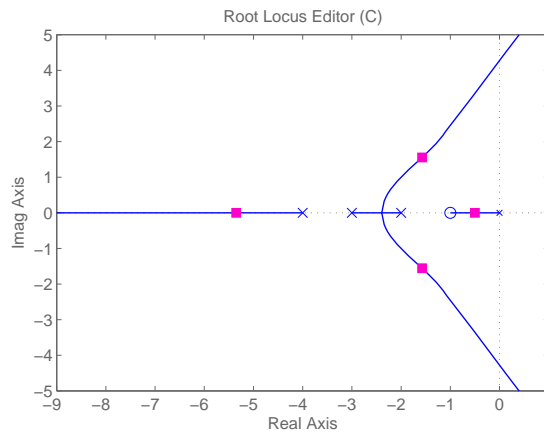
- d. The value of gain for which the closed-loop transfer function will have a pole on the real axis at -0.5 .

If a pole is on the real axis at -0.5 , then it should be on the root locus, which means that $s = -0.5$ should satisfy the following relation:

$$1 + \frac{K(s+1)}{s(s+2)(s+3)(s+4)} \Big|_{s=-0.5} = 0.$$

Hence,

$$K = \frac{-(0.5)(1.5)(2.5)(3.5)}{0.5} = 13.12.$$



Note that one pink dot is indeed at $s = -0.5$ for this value of the gain $K = 13.12$.

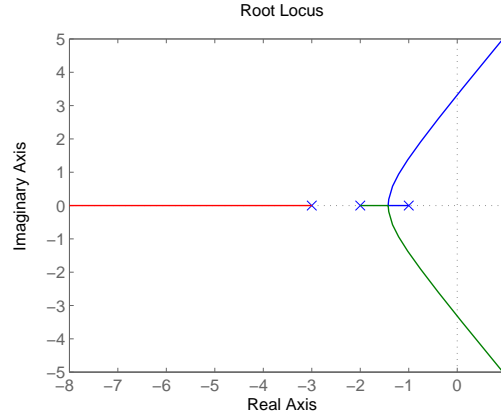
5. Nise problem 24 from chapter 8, page 481.

Answer:

The open-loop transfer function is

$$G(s) = \frac{K}{(s+1)(s+2)(s+3)}.$$

a. Sketch the root locus.



b. K for 20% overshoot

To find K for 20% overshoot, we have to identify the location of poles that generates 20% overshoot. Then using the values of the poles and $1 + KG(s) = 0$, we can compute the gain K at the poles.

For the first step, we have to find the location of the poles which produce 20% overshoot. The damping ratio ζ determines how much overshoot is generated. You can estimate the damping ratio ζ from the root locus, using the relation $\zeta = \cos \theta$, where θ is the angle subtended from the pole to the origin of the s -plane. (Please refer to pp.2 of lecture note 20.) Therefore, the poles that yield 20% overshoot should be the straight line $\cos \theta = \zeta$ for the value of ζ that yields 20% OS.

From the relation between %OS and damping ratio, we find

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.4559.$$

Also, $\cos \theta = \zeta \Rightarrow \theta = \pm 62.9^\circ$.

Next step is to find the location of the pole at $\zeta = 0.4559$. MATLAB's `sisotool` provides you damping ratio, pole location, and gain K . (Also it gives the natural frequency ω_n , which can be used in the next problem.) It turns out that $\zeta = 0.4559$, $p = -0.866 + \pm j1.691$, and $K = 9.4037$. You will find approximate values due to tuning error of the gain.

If you want to find the exact location of the pole analytically, you can try as follows: Let's assume that a pole $p = -\sigma + j\omega$. From the damping ratio,

$\theta = 62.9^\circ$, which means $\omega = \tan(62.9^\circ) \times \sigma = 1.954\sigma$. To be on the root locus, the pole $-\sigma + j\omega$ should satisfy $1 + KG(-\sigma + j\omega) = 0$. We expand this equation as follows:

$$s^3 + 6s^2 + 11s + (6 + K) = 0.$$

$$(-\sigma + j\omega)^3 + 6(-\sigma + j\omega)^2 + 11(-\sigma + j\omega) + (6 + K) = 0.$$

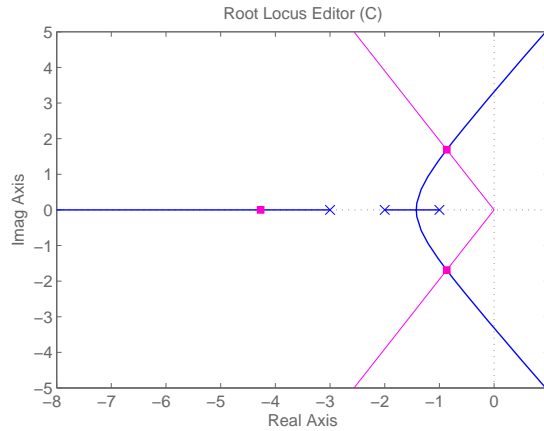
$$(-\sigma^3 + 3\sigma^2j\omega + 3\sigma\omega^2 - \omega^3j) + 6(\sigma^2 - 2\sigma\omega j - \omega^2) - 11\sigma + 11\omega j + 6 + K = 0.$$

Separating real and imaginary parts as

$$-\sigma^3 + 3\sigma\omega^2 + 6\sigma^2 - 6\omega^2 - 11\sigma + 6 + K = 0,$$

$$3\sigma^2\omega - \omega^3 - 12\sigma\omega + 11\omega = 0,$$

and using $\omega = 1.954\sigma$, we find $\sigma = 0.866$ and $\omega = 1.691$ from the second equation. Using these values in the first equation, we find $K = 9.4037$.

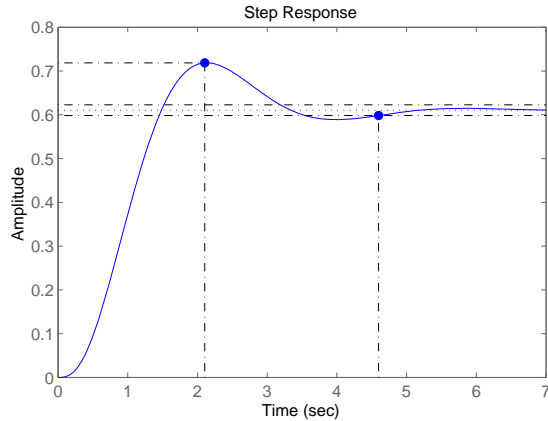


Note that the pink line indicates $\zeta = 0.4559$, $\cos \theta = \zeta$ ($\theta = \pm 62.9^\circ$).

- c. The settling time, peak time for K found in (b)

To estimate the settling time and peak time, we have to know the damping ratio and natural frequency. From the previous question, we found the damping ratio ζ (and natural frequency ω_n as well if you used MATLAB `sisotool`). If you solved (b) analytically, you know the pole $p = -0.866 \pm j1.691$. The imaginary part of the pole corresponds the damped natural frequency ω_d , which is $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. Hence, the natural frequency $\omega_n \approx 1.9$ (rad/s).

- The settling time $T_s \approx 4/(\zeta\omega_n) = 4.6$ (s).
- The peak time $T_p = \pi/(\omega_n\sqrt{1 - \zeta^2}) \approx 2.1$ (s).



Note that the setting time and peak time computed by the second order approximation well agree with the MATLAB result plotted above.

- d. The locations of higher-order poles for K found in (b)

If the pole is purely real, $1 + KG(s) = 0$ for $s = \sigma$. Solving

$$1 + (\sigma + 1)(\sigma + 2)(\sigma + 3) = -9.4037$$

numerically, we find $\sigma = -4.2681$. Since $|-4.2618| \approx 5 \times |-0.866|$, therefore the second order assumption is valid in this case.

- e. The range of K for stability

We find the imaginary axis crossings by setting $KG(j\omega) = -1$ (rule 7). Solving

$$-\omega^3 j - 6\omega^2 + 11\omega j + 6 + K = 0,$$

and separating to real and imaginary parts respectively,

$$\begin{aligned} -\omega^3 + 11\omega &= 0, \\ -6\omega^2 + 6 + K &= 0, \end{aligned}$$

we find $\omega^2 = 11$ and hence $K = 6 \times 11 - 6 = 60$. At $K = 60$, the system is marginally stable and for $K > 60$ it becomes unstable.