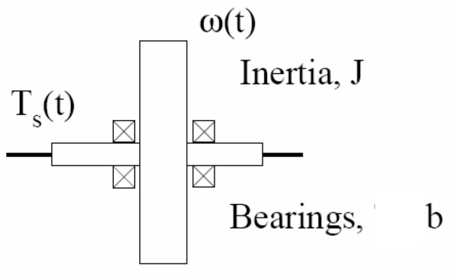
Review: step response of 1st order systems we've seen

- Inertia with bearings (viscous friction)

Step input $T_s(t) = T_0 u(t) \Rightarrow$ Step response*

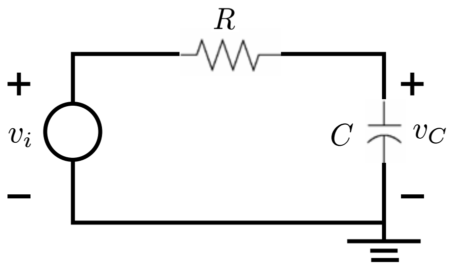
$$\omega(t) = \frac{T_0}{b}\left(1 - \mathrm{e}^{-t/\tau}\right), \quad \text{where} \quad \tau = \frac{J}{b}.$$



- RC circuit (charging of a capacitor)

Step input $v_i(t) = V_0 u(t) \Rightarrow$ Step response

$$v_C(t) = V_0\left(1 - \mathrm{e}^{-t/\tau}\right), \quad \text{where} \quad \tau = RC.$$

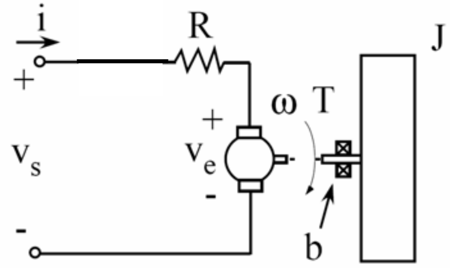


- DC motor with inertia load, bearings and negligible inductance

Step input $v_s(t) = V_0 u(t) \Rightarrow$ Step response

$$\omega(t) = \frac{K_m}{R} V_0 \left(1 - \mathrm{e}^{-t/\tau}\right),$$

$$\text{where} \quad \tau = \frac{J}{\left(b + \dfrac{K_m K_v}{R}\right)}.$$

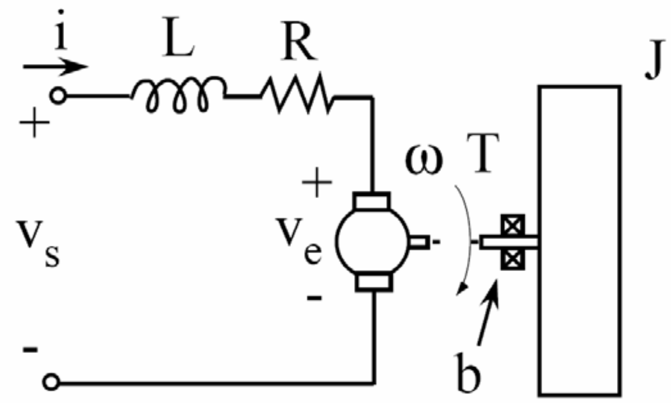


*When writing the step response, we may omit the step function $u(t)$ implying the result holds for $t > 0$ only.

Goals for today

- First-order systems response
  - pole, zero definitions
  - the significance of poles and zeros:
    from *s*-domain representation to transient characteristics
- DC motor dynamics:
  - angular velocity (1^{st} order: 1 pole)
  - current (1^{st} order: 1 pole, 1 zero)
- Two new properties of the Laplace transform:
  - final value theorem
  - initial value theorem
- **Next two lectures:**
  - 2^{nd} order systems

Current dynamics in DC motor system



Recall **combined equations of motion**

$$\left.\begin{aligned} LsI(s) + RI(s) + K_v\Omega(s) &= V_s(s) \\ Js\Omega(s) + b\Omega(s) &= K_m I(s) \end{aligned}\right\} \Rightarrow$$

$$\begin{cases} \left[\dfrac{LJ}{R}s^2 + \left(\dfrac{Lb}{R} + J\right)s + \left(b + \dfrac{K_m K_v}{R}\right)\right]\Omega(s) = \dfrac{K_m}{R}V_s(s) \\ (Js + b)\,\Omega(s) = K_m I(s) \end{cases}$$

Neglecting the DC motor's inductance (*i.e.*, assuming $L/R \approx 0$), we find

$$\begin{cases} \dfrac{\Omega(s)}{V_s(s)} = \dfrac{\dfrac{K_m}{RJ}}{s + \dfrac{1}{J}\left(b + \dfrac{K_m K_v}{R}\right)} \\[2ex] \dfrac{I(s)}{V_s(s)} = \dfrac{\dfrac{1}{R}\left(s + \dfrac{b}{J}\right)}{s + \dfrac{1}{J}\left(b + \dfrac{K_m K_v}{R}\right)} \end{cases}$$

Transfer function for the angular velocity is of the form

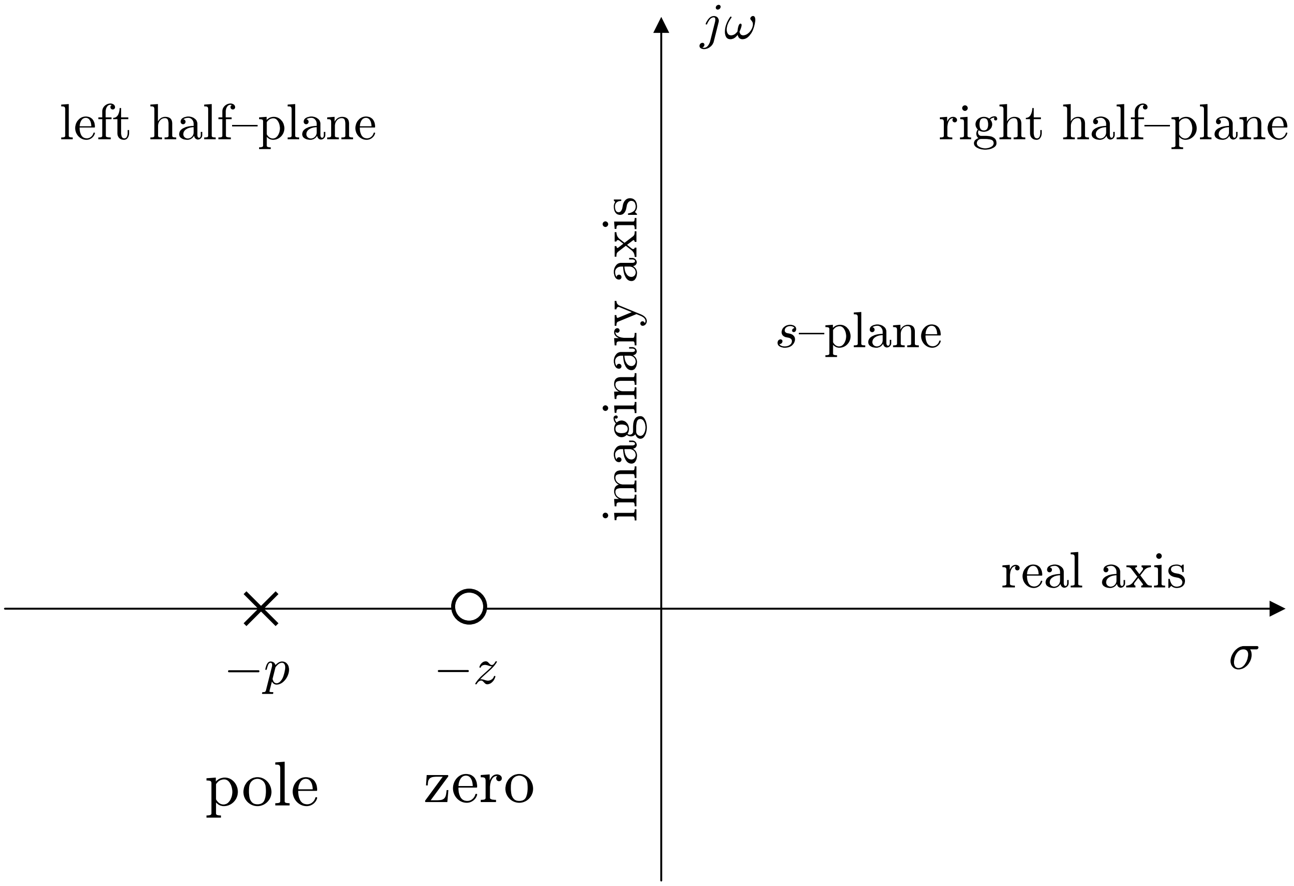
$$\frac{A}{s + \boxed{p}}, \qquad p = \frac{1}{J}\left(b + \frac{K_m K_v}{R}\right) \leftarrow \text{pole}$$

Transfer function for the current is of the form

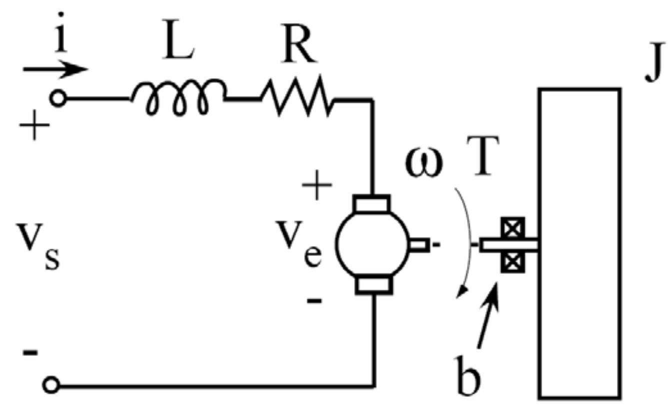
$$\frac{B\,(s + \boxed{z})}{s + \boxed{p}}, \qquad p = \frac{1}{J}\left(b + \frac{K_m K_v}{R}\right), \qquad z = \frac{b}{J}$$

zero

Poles and zeros in the *s*-plane



DC motor step response: numerical example



$$L \approx 0,\ R = 6\Omega,\ K_v = 6\text{V}\cdot\text{sec},$$
$$K_m = 6\text{N}\cdot\text{m/A},\ J = 2\text{kg}\cdot\text{m}^2,\ b = 4\text{kg}\cdot\text{m}^2\cdot\text{Hz}.$$

We will compute the system's response (both angular velocity and current) to the step input $v_s(t) = 30u(t)$ V.

Substituting the numerical values into the system TF,

whereas the Laplace transform of the input is

$$V_s(s) \equiv \mathcal{L}\left[v_s(t)\right] = \mathcal{L}\left[30u(t)\right] = \frac{30}{s}.$$

we find

$$\begin{cases} \dfrac{\Omega(s)}{V_s(s)} = \dfrac{1}{2}\dfrac{1}{s+5\,[\text{Hz}]} \\[2ex] \dfrac{I(s)}{V_s(s)} = \dfrac{1}{6}\dfrac{(s+2\,[\text{Hz}])}{s+5\,[\text{Hz}]} \end{cases}$$

I. Angular velocity

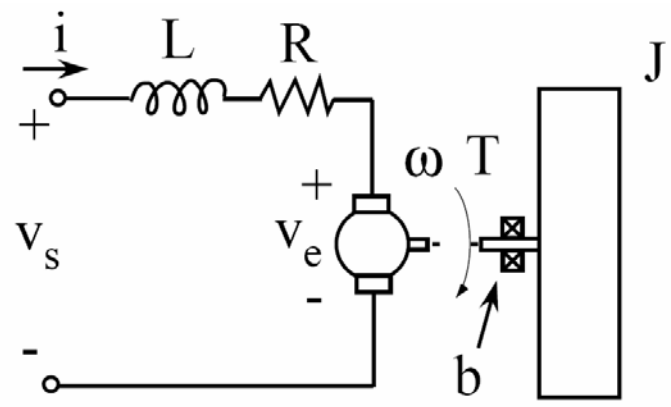
$$\Omega(s) = \frac{15}{s\,(s+5)} = 15\left(\frac{K_1}{s} + \frac{K_2}{s+5}\right) \quad \text{where} \quad K_1 = \left.\frac{1}{s+5}\right|_{s=0} = \frac{1}{5},\ K_2 = \left.\frac{1}{s}\right|_{s=-5} = -\frac{1}{5} \Rightarrow$$

$$\Omega(s) = 3\left(\frac{1}{s} - \frac{1}{s+5}\right) \quad \Rightarrow \quad \omega(t) = 3\left(1 - \mathrm{e}^{-5t}\right)u(t)\ \frac{\text{rad}}{\text{sec}}.$$

Forced response

Homogeneous response **(Natural response)**

DC motor step response: numerical example



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II. Current

$$I(s) = \frac{5\,(s+2)}{s\,(s+5)} = 5\left(\frac{K_1'}{s} + \frac{K_2'}{s+5}\right) \quad \text{where} \quad K_1' = \left.\frac{s+2}{s+5}\right|_{s=0} = \frac{2}{5},\quad K_2' = \left.\frac{s+2}{s}\right|_{s=-5} = \frac{3}{5} \Rightarrow$$

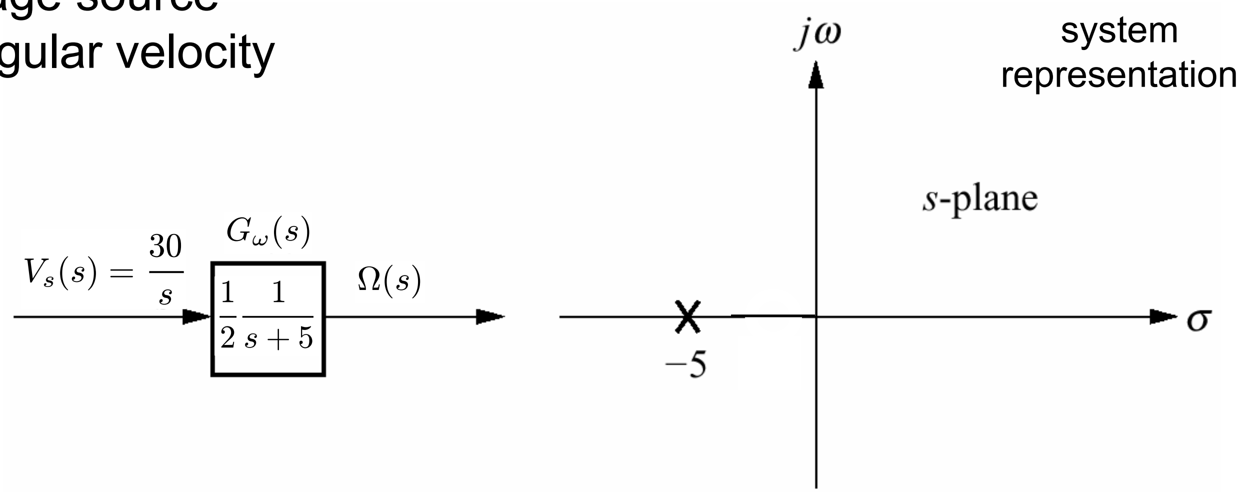
$$I(s) = \left(\frac{2}{s} + \frac{3}{s+5}\right) \quad \Rightarrow \quad i(t) = \left(2 + 3\mathrm{e}^{-5t}\right)u(t) \quad \text{A}.$$

Forced response

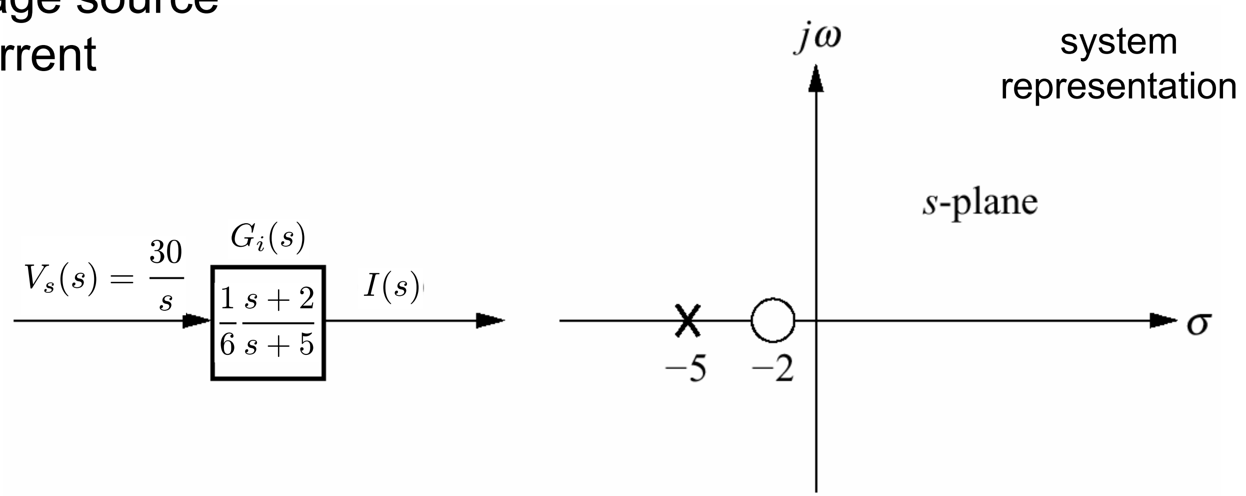
Homogeneous response
(Natural response)

DC motor system in the *s*-plane

Input: Voltage source
Output: Angular velocity



Input: Voltage source
Output: Current



DC motor step response (angular velocity)

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Please see: Fig. 4.1 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

DC motor step response (current)

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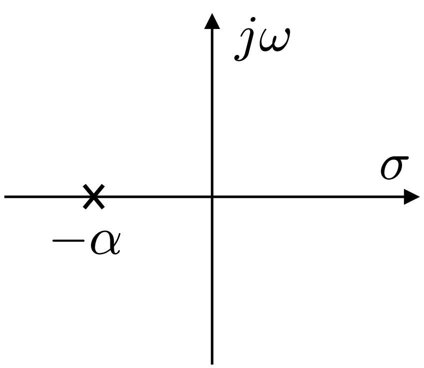
Please see: Fig. 4.1 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

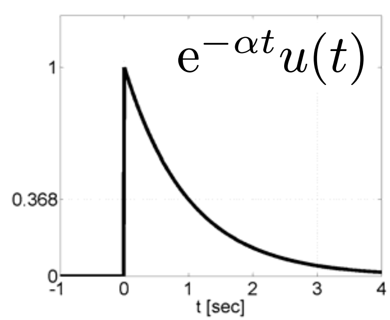
1st order system response from *s*-plane representation

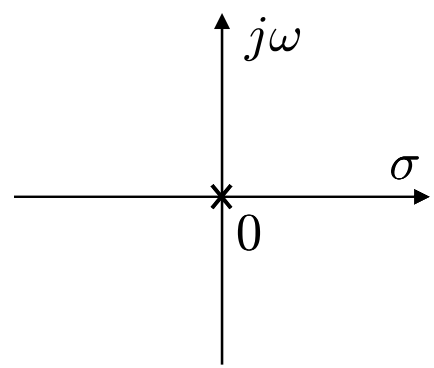
- Pole in the input function generates forced response
- Pole in the transfer function generates natural response
- Zero in the transfer function does not alter the speed of settling to steady state (i.e. the time constant) but it does alter the relative amplitudes of the forced and natural responses

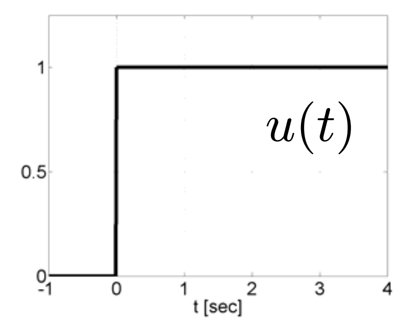
- Pole at $-\alpha$ generates response $e^{-\alpha t}$ (exponentially decreasing if pole on the right half-plane; increasing if on the left half-plane)

- Pole at zero generates step function









Time constant $\tau=1/\alpha$

1^{st} order system: transient response properties

Step response in the s–domain

$$\frac{a}{s(s+a)};$$

in the time domain

$$\left(1-e^{-at}\right)u(t);$$

time constant

$$\tau = \frac{1}{a};$$

rise time (10%→90%)

$$T_r = \frac{2.2}{a};$$

settling time (98%)

$$T_s = \frac{4}{a}.$$

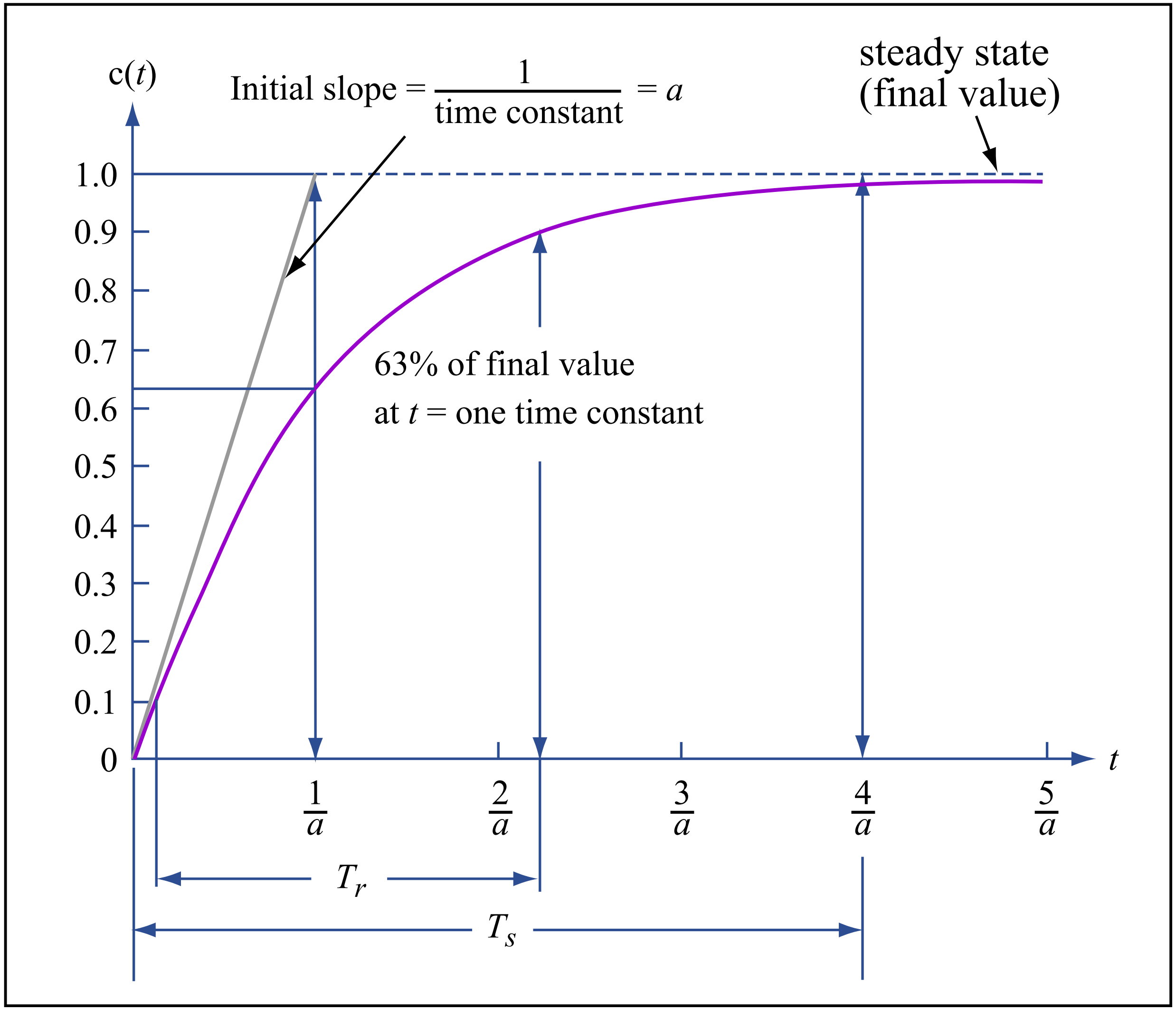


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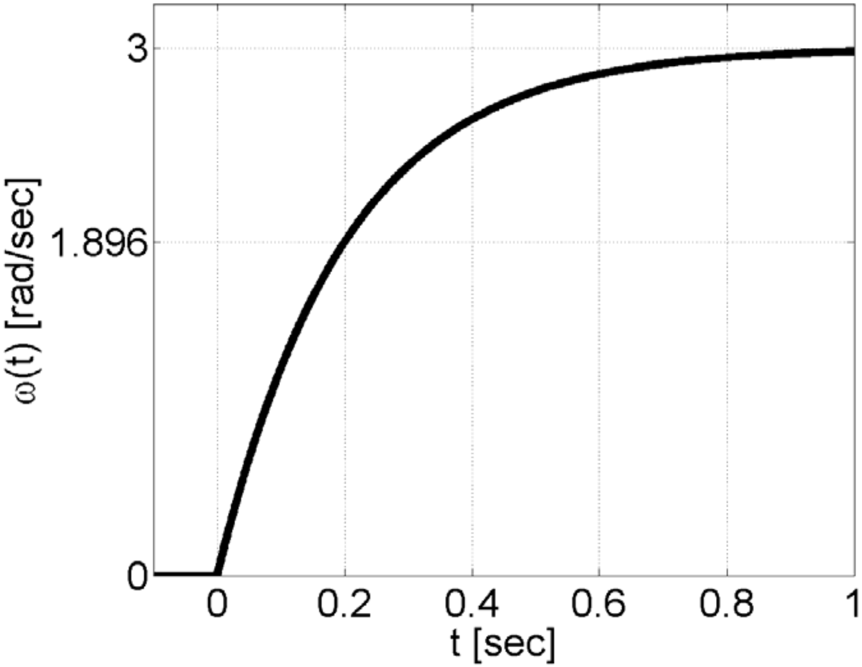
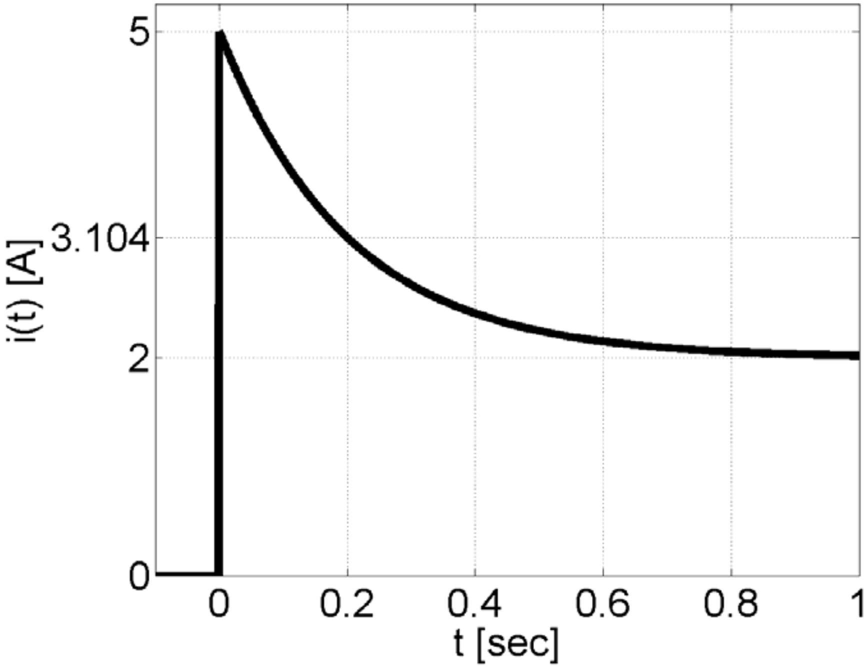
Figure 4.3

DC motor step responses





$$\omega(t) = 3\left(1 - \mathrm{e}^{-5t}\right)u(t) \quad \mathrm{rad/sec}$$

$$\omega(\infty) = 3 \quad \mathrm{rad/sec}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t}(0+) = 3 \times 5 = 15 \quad \mathrm{rad/sec^2}$$

$$i(t) = \left(2 + 3\mathrm{e}^{-5t}\right)u(t) \quad \mathrm{A}$$

$$i(\infty) = 2 \quad \mathrm{A}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t}(0+) = \infty$$

The Final Value theorem: steady-state

We will now learn two additional properties of the Laplace transform, which we will quote without proof. Let $F(s)$ denote the Laplace transform of the function $f(t)$. The first property is the

Final Value theorem:

$$f(\infty) = \lim_{s\to 0} sF(s);$$

Let us see how this applies to the step response of a general 1^{st}–order system with a pole at $-a$ and without a zero (e.g., the angular velocity response of the DC motor.) We select the system gain such that the steady–state will equal 1. The step response in the s–domain then is

$$F_1(s) = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}; \qquad \text{also,} \qquad sF_1(s) = \frac{a}{s+a}.$$

Using the partial fraction expansion above, we find the time domain step response as

$$f_1(t) = \left(1 - \mathrm{e}^{-at}\right) u(t) \Rightarrow f_1(\infty) = 1 \qquad \text{(as advertised.)}$$

Applying the final value theorem, we find, consistently,

$$f_1(\infty) = \lim_{s\to 0} sF_1(s) = \lim_{s\to 0} \frac{a}{s+a} = 1.$$

The Initial Value theorem: initial slope

The second property of the Laplace transform is the

Initial Value theorem

$$f(0+) = \lim_{s\to\infty} sF(s).$$

Let us use this property to compute the initial slope of the step response, *i.e.* the value of the derivative of the step response at $t = 0+$ for the same general 1^{st}–order system with steady state equal to unity, a pole at $-a$ and without a zero. Since we are interested in the derivative of $f(t)$, the Laplace transform of interest is

$$H_1(s) = \mathcal{L}\left[\frac{\mathrm{d}f_1(t)}{\mathrm{d}t}\right] = sF_1(s) \Rightarrow sH_1(s) = s^2F_1(s) = \frac{as}{s+a}.$$

Applying the final value theorem,

$$\frac{\mathrm{d}f_1}{\mathrm{d}t}(0+) = \lim_{s\to\infty}\frac{as}{s+a} = \lim_{s\to\infty}\frac{a}{1+a/s} = a.$$

Again, this is consistent with the result we get directly from the time domain:

$$f_1(t) = \left(1 - \mathrm{e}^{-at}\right) \Rightarrow \frac{\mathrm{d}f_1}{\mathrm{d}t}(t) = -\left(-a\right)\mathrm{e}^{-at} \Rightarrow \frac{\mathrm{d}f_1}{\mathrm{d}t}(0+) = a.$$

Initial and final value of 1st-order system with a zero

We now consider the same 1^{st}–order system with unity steady state, a pole at $-a$, but we also add a zero at $-z$. In that case, the Laplace transforms of the step response and its derivative are

$$F_2(s) = a\frac{s+z}{s(s+a)} = \frac{z}{s} + \frac{a-z}{s+a}; \qquad sF_2(s) = a\frac{s+z}{s+a}; \qquad s^2F_2(s) = a\frac{s(s+z)}{s+a}.$$

We can readily see that

$$f_2(\infty) = \lim_{s\to 0} sF_2(s) = z; \qquad f_2(0+) = \lim_{s\to\infty} sF_2(s) = a; \qquad \frac{\mathrm{d}f_2}{\mathrm{d}t}(0+) = \lim_{s\to\infty} s^2F_2(s) = \infty.$$

You should verify that these results are consistent with the time–domain solution for this system.

We can see that the effects of the zero $-z$ on the 1^{st}–order system are (in comparison to a system with the same pole at $-a$ but without the zero)

- amplify the steady–state response by z;
- raise the initial value from zero to A;
- raise the initial slope to infinity.

The infinite initial slope is non–physical; in the case of the DC motor, it occurs because we neglected the inductance L.

How the zero acts

Comparing the system response $F_1(s)$ (without a zero) and the system response $F_2(s)$ (with a zero at $-z$), we can see that

$$F_2(s) = (s+z)F_1(s) = sF_1(s) + zF_1(s) \Rightarrow$$

$$f_2(t) = \frac{\mathrm{d}f_1(t)}{\mathrm{d}t} + zf_1(t).$$

That is, the zero results in **derivative action** and **amplification**.
Both of these results are qualitatively consistent with our observations from the previous page.

DC motor