Integration and differential equations

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ISBN 978-87-7681-970-5

Contents

Preface to these two texts

These two texts in this one cover, entitled 'An introduction to the standard methods of elementary integration' (Part I) and 'The integration of ordinary differential equations' (Part II), are two of the 'Notebook' series available as additional and background reading to students at Newcastle University (UK). This pair constitutes a basic introduction to both the elementary methods of integration, and also the application of some of these techniques to the solution of standard ordinary differential equations. Thus the first is a natural precursor to the second, but each could be read independently.

Each text is designed to be equivalent to a traditional text, or part of a text, which covers the relevant material, with many worked examples and some set exercises (for which the answers are provided). The necessary background is described in the preface to each Part, and there is a comprehensive index, covering the two parts, at the end.

Part I

An introduction to the standard methods of elementary integration

List of Integrals

This is a list of the integrals, and related problems, that constitute the worked examples considered here.

$y = f(x) = x^2$ (Riemann integral)………... **p.14;** $\int_0^1 e^{3x} dx$ …..…… **p.19;**

primitives of $(2x+1)^5, (3-4x)^{3/2}, (3x+4)^{-1}, 2\sin(17x-6), \tan(1-2x), \frac{3}{x^2+16}$,

$\frac{5}{\sqrt{x^2-2x-8}}$, $x^2\sin(x^3)$, $\frac{1+2x}{x^2+x-3}$ **all p.25**;

$\int_1^2 \left[x^2+(3x-1)^{-2}\right]dx$, $\int_0^{1/4} 3\sec^2\left(\frac{\pi}{2}-x\pi\right)dx$, $\int_1^2 \frac{dx}{\sqrt{9-x^2}}$ ………**all p.26;**

$\int x^n \ln x\, dx$ ($x>0$, $n \neq -1$), $\int \arctan x\, dx$, $\int_0^1 x^2 e^{-x} dx$, $\int_0^2 \sqrt{4-x^2}\, dx$ ....**all p.27;**

$\int (1+\ln x)^n dx$ ($x>0$, $n \neq -1$), $\int_1^2 (1+\ln x)^3 dx$ …..**both p.30;** $I_n = \int \tan^n \theta\, d\theta$ ($n \geq 2$)

$\int \tan^6 \theta\, d\theta$ …..**both p.32;** do $\int_0^\infty \frac{dx}{1+x^2}$, $\int_0^\infty \sin x\, dx$, $\int_0^\infty \frac{dx}{1+x}$ exist?…………….**all p.33;**

do $\int_0^1 \frac{dx}{\sqrt{x}}$, $\int_{-1}^2 \frac{dx}{1+x}$, $\int_0^1 \ln x\, dx$ exist?………………**all p.35;** arctan(sinh x), 2 arctan(e^x),

2 arctan(tanh(x/2)), arcsin(tanh x) all primitives of $\int \operatorname{sech} x\, dx$ ……………..…..…**p.36;**

divide $\frac{x^5-3x^4+5x^3-2x^2+x-1}{x^2+x+1}$ ………**p.38;** $\int \frac{x^3-x^2+2x+1}{x+1} dx$ ………....**p.39;**

$\int \frac{x^6-3x^5+2x^4+x^3-x^2+4x-7}{(x-2)^3} dx$ ………**p.40;** $\int \frac{x^3-1}{x^2+4x+3} dx$ ………..…..**p.41;**

$\int_1^2 \frac{x^4-2x^2+3x-2}{x^2-6x+9} dx$ ....**p.42;** $\int \frac{x^3-3x+1}{x^2-8x+25} dx$ ….**p.43;** $\int \frac{x^6-x^3+1}{x^3+x^2+x+1} dx$ …..**p.44;**

$\int \frac{x-2}{(x+1)(x^2-1)(x^2+2x+2)(x^2+2x+5)} dx$ …**p.45;** $\int \frac{x^2-x+1}{(x-3)(x^2+4x+13)^2} dx$ ..**p.48;**

$\int \sin 2x \cos 3x\, dx$ ……………..**p.50;** $\int_0^{\pi/2} (\sin 2x - \sin 4x)\sin 2x\, dx$ …………....…..**p.50;**

$\int \sin^{10} x \cos^5 x\, dx$ ……….**p.51;** $\int_0^{\pi/2} \sin^5 x \cos^2 x\, dx$ ……....**p.52;** $\int \sin^4 x\, dx$ ……..**p.53;**

$\int \frac{dx}{5+4\cos x}$ …………..…………….**p.54;** $\int \frac{\sin^2 x}{(2+\cos x)^2} dx$ ………..………….**p.55**.

Preface

The material presented here is intended to provide an introduction to the methods for the integration of elementary functions. This topic is fundamental to many modules that contribute to a modern degree in mathematics and related studies – and especially those with a mathematical methods/applied/physics/engineering slant. The material has been written to provide a general introduction to the relevant ideas, and not as a text linked to a specific course of study that makes use of integration. Indeed, the intention is to present the material so that it can be used as an adjunct to a number of different modules or courses – or simply to help the reader gain a broader experience of, and an improved level of skill in, these important mathematical ideas. The aim is to go a little beyond the routine methods and techniques that are usually presented in conventional modules, but all the standard ideas are discussed (and can be accessed through the comprehensive index). This material should therefore prove a useful aid to those who want more practice, and also to those who would value a more complete and systematic presentation of the standard methods of integration.

It is assumed that the reader has a basic knowledge of, and practical experience in, the methods of integration that are typically covered at A-level (or anything equivalent), although the presentation used here ensures that the newcomer should not be at a loss. It is, however, necessary that the reader has met the simplest ideas (and results) that underpin the differential calculus, and to have met partial fractions. This brief notebook does not attempt to include any applications of integration; this is properly left to a specific module that might be offered, most probably, in a conventional applied mathematics or engineering mathematics or physics programme. However, there is much to be gained by a more formal 'pure mathematical' introduction to these ideas, and so we will touch on this at the start of Chapter 1. Some of the methods are employed in the integration of ordinary differential equations, which comprises Part II of this text.

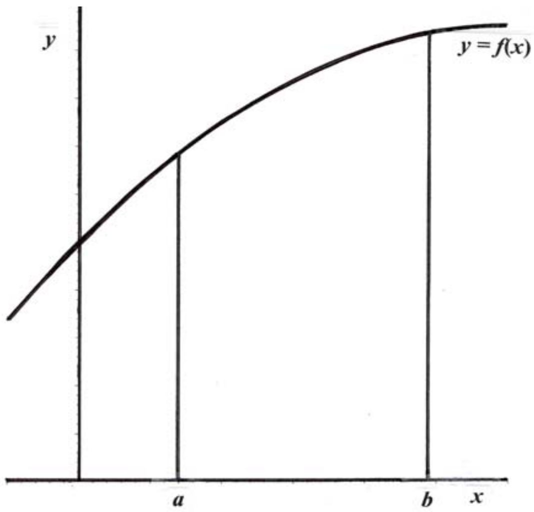
The approach adopted here is to present some general ideas, which initially involve a notation, a definition, a theorem, etc., in order to provide a fairly firm foundation for the techniques that follow; but most particularly we present many carefully worked examples (there are 45 in total). Some exercises, with answers, are also offered as an aid to a better understanding of the methods.

1 Introduction and Background

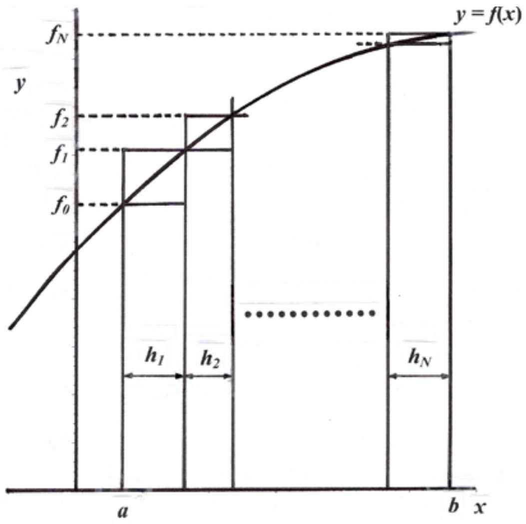
Being able to integrate functions is an essential skill for a mathematician (or engineer or physicist) who has an interest in, for example, mathematical methods and applications – or who just likes a challenge! Certainly many simple integration problems can be answered by a mathematical package (such as Maple), but even this – on occasion – fails if the function to be integrated has not been written in an appropriate form. (It also fails to integrate functions that, typically, involve an arbitrary power and require the construction of a recursion formula.) In addition, Maple will produce only one form of the answer, even when others are available (and we may prefer one rather than another, in a particular context). Here, we will devote much effort to a systematic development of the basic techniques of integration. However, we shall begin by laying appropriate foundations, which will include a brief description of the notion of integration and its connection with the differential calculus: the *fundamental theorem of calculus*. We will then use this theorem as the springboard for obtaining the basic rules of integration and for deriving, for example, the method of integration by parts. Of course, we shall include a careful discussion of both definite and indefinite integrals.

1.1 The Riemann integral

We start with the familiar problem of finding the area under a (continuous and bounded) curve. Let the curve be $y = f(x)$, and suppose that the area required lies between $x = a$ and $x = b$ ($b > a$) and between $y = 0$ and $y = f(x)$; this is shown in the figure below



where, for convenience, we have drawn the case $f(x) > 0$. (We will discuss the situation where $f(x) < 0$ a little later.) The area under the curve is estimated by rectangles ('slices', if you like) of width h_1, h_2, …, h_N, and each of height $f_n = f(x_n)$, $n = 0, 1, \ldots, N$, where $x = x_n$ is some value in (or on the boundary of) the rectangle of width h_n. Furthermore, this height of the rectangle can be chosen to produce estimates of the area that are either above or below the actual area; an example of this is shown in the figure below, where the maximum and minimum values in a particular rectangle are easily identified:



Thus the area under the curve can be approximated by

$$f(x_0)h_1 + f(x_1)h_2 + \ldots . f(x_{N-1})h_N = \sum_{n=1}^{N} f(x_{n-1})h_n$$

or, equivalently, $\sum_{n=1}^{N} f(x_n)h_n$.

Definition

If $\sum_{n=1}^{N} f(x_n)h_n$ has a limit as $h_n \to 0$ ($n = 1,2,\ldots,N$, with $N \to \infty$), no matter how the region is subdivided, then this is called the *Riemann integral* of $f(x)$. (Equally, we could have used $\sum_{n=1}^{N} f(x_{n-1})h_n$.) We write the limit as $\int_a^b f(x)\,\mathrm{d}x$.

This notation for the integral comes from the shape of the 's' (for 'sum') used in the 17th century – the $\int$ symbol – and the width of the subdivision, originally $'\delta x'$. Note that this definition used by Riemann has produced the *definite integral*. Before we proceed, it is instructive to see that this approach can be used to obtain the integral 'directly', without recourse to conventional integration.

Example 1

Use Riemann's definition of the integral to find the area under the curve

$$y = f(x) = x^2 \text{ from } x = 0 \text{ to } x = X\ (>0).$$

We choose to construct subdivisions of equal width (and since any is permitted, this is allowed), so an approximation to the area is

$$f(h)h + f(2h)h + \ldots.. f(Nh)h$$

where we have $Nh = X$. Thus we obtain

$$\text{area} \approx \sum_{n=1}^{N} h(nh)^2 = h^3\left(1^2 + 2^2 + 3^2 + \ldots. + N^2\right)$$

$$= h^3.\tfrac{1}{6}N(N+1)(2N+1)$$

$$= \tfrac{1}{6}X(X+h)(2X+h).$$

Now the exact expression for the area is obtained from

$$A = \lim_{h\to 0}\left\{\tfrac{1}{6}X(X+h)(2X+h)\right\} \text{ for fixed } X$$
$$= \tfrac{1}{3}X^3.$$

[G.F.B. Riemann (1826-1866), German mathematician, laid the foundations for Einstein's work by developing the theories of non-Euclidean geometry; he also made very significant contributions to topological spaces and to analysis.]

Comment: If we wished, we could make this process more mathematically robust by showing that the limit (the area) does exist and so avoid the notion of 'approximately equal to'. Thus we may write (see the figure above)

$$\sum_{n=1}^{N} h\left[(n-1)h\right]^2 < A < \sum_{n=1}^{N} h(nh)^2$$

for any finite *N*; this gives

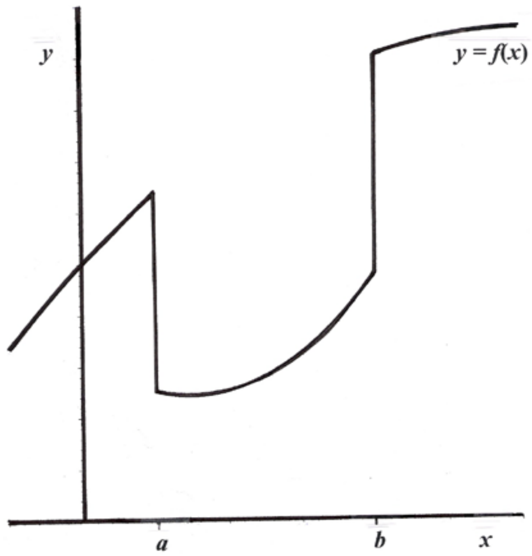
$$\tfrac{1}{6}h^3(N-1)N(2N-1) < A < \tfrac{1}{6}h^3 N(N+1)(2N+1)$$

or $$\tfrac{1}{6}(X-h)X(2X-h) < A < \tfrac{1}{6}X(X+h)(2X+h).$$

Now as $h \to 0$, for *X* fixed, this confirms that $A \to \tfrac{1}{3}X^3$.

One immediate consequence of the Riemann definition is that if $f(x) < 0$, we shall produce a negative value for $f \times h$ – and so we have the notion of 'negative area'. This, as we can readily see, is simply the result of computing an area when the curve $y = f(x)$ drops below the *x*-axis; if we then wanted the 'physical' area, we would form $-\sum f(x_n)h$ for every $f(x_n) < 0$. Of course, if some $f(x_n) > 0$ and some $f(x_n) < 0$, we shall be adding positive and negative contributions, and some cancellation will inevitably occur. It is clear, therefore, that if we wanted the physical area *between* the curve and the *x*-axis, we must first identify the points where $y = f(x)$ crosses the axis, and then compute the Riemann integral between each consecutive pair of zeros. The negative areas are then multiplied by -1 and added to the positive-valued contributions.

We observe that another significant consequence of the Riemann definition is that we can permit the function, $f(x)$, to be discontinuous (provided that the area remains finite); thus we may allow the type of function represented in the figure below:



That this is possible, within the definition, is immediately apparent if we choose boundaries between particular subdivisions to coincide with $x = a$ and with $x = b$ (in the figure). (Those already familiar with elementary integration will recognise here the equivalent interpretation of integrating up to $x = a$, then between $x = a$ and $x = b$, and then from $x = b$, and adding the results.)

Finally, we note that we may disconnect the definition of the Riemann integral from the concept of area. It is sufficient to be given $y = f(x)$, and a subdivision from $x = a$ and $x = b$ into widths h_1, h_2, …, with associated values x_1, x_2, … . Then we form the object

$$\sum_{n=1}^{N} f(x_n)h_n \text{ ;}$$

if this has a limit as $h_n \to 0$, $n = 1,2,\ldots,N$ (so $N \to \infty$), then we write it as $\int_a^b f(x)\,\mathrm{d}x$.

It is clear that, at this stage, we have not introduced the concept of 'integration' in the calculus sense (as will probably be familiar from school mathematics), nor do we have the 'indefinite integral'. All this hinges on the next important – indeed fundamental – idea that we must introduce.

Historical Note: The technique of putting upper and lower bounds on an area, and then using some limiting argument – or though it was never expressed like this – was known to the Greek mathematicians, in particular Eudoxus (408-355BC), and exploited with great success by Archimedes (287-212BC). (So Riemann's definition amounts to a more precise statement of the ideas developed well over 2000 years earlier.) Methods for finding the area under a curve were rather well-developed by the time of Newton (1642-1727) and Leibniz (1646-1716), but these methods tended to be rather *ad hoc*, tailored to specific types of problem. The fundamental connection with the differential calculus (see below) was made – altogether independently – by Newton and Leibniz, although it was the latter who gave us the name 'integral calculus'. He wanted to convey the idea of *integrate* in the sense of 'bringing together' or 'combining'; that is, the summing ('combining') of the elements ('slices') that contribute to the total area. Hence Leibniz introduced the word *integral* to be the total area, and then the link with the differential calculus followed naturally – for Leibniz – to the name *integral calculus*.

1.2 The fundamental theorem of calculus

Let us consider the function defined by

$$A(X) = \int_a^X f(x)\,\mathrm{d}x \quad (X > a)$$

which we may think of as the (signed) area under the curve $y = f(x)$, from $x = a$ to $x = X$, if that is helpful. Further, we introduce

$$A(X+h) = \int_a^{X+h} f(x)\,\mathrm{d}x$$

and then we form

$$A(X+h) - A(X) = \int_a^{X+h} f(x)\,\mathrm{d}x - \int_a^X f(x)\,\mathrm{d}x \text{ ;;}$$

but from the Riemann definition, we have that

$$A(X+h) - A(X) \approx f(X + \theta h)h$$

where $0 \le \theta \le 1$ is arbitrarily assigned to give a suitable approximate value – or even the exact value. (We assume, for convenience here, that $f(x)$ is continuous for $a < x < X$.) Thus we have

$$\frac{A(X+h) - A(X)}{h} \approx f(X + \theta h)$$

which becomes progressively more accurate as $h \to 0$; in the limit we obtain

$$\lim_{h \to 0} \left[\frac{A(X+h) - A(X)}{h} \right] = f(X)$$

$$\text{i.e. } \frac{\mathrm{d}A}{\mathrm{d}X} = f(X).$$

i.e. $\frac{\mathrm{d}A}{\mathrm{d}X} = f(X)$.

Thus the function $A(X)$ – the integral of $f(X)$ – is that function whose derivative is $f(X)$: it is the 'anti-derivative' of $f(X)$. (Note that, in this context, the function must also satisfy $A(a) = 0$.) Hence the problem of finding $\int_a^b f(x)\,\mathrm{d}x$ is one of reversing the process of differentiation: we have introduced the essential connection between the differential calculus and the fundamental notion of integration.

1.3 Primitives

Given any function $f(x)$, the construction of a function whose derivative is $f(x)$ produces a *primitive*. So, for example, given $f(x) = x^2$, we have that $\frac{1}{3}x^3$ is a primitive, but so is $\frac{1}{3}x^3 + 1$ and $\frac{1}{3}x^3 - 1000$, because for each

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{3}x^3\right) = x^2\,;\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{3}x^3 + 1\right) = x^2\,;\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{3}x^3 - 1000\right) = x^2,$$

according to the elementary rules of the differential calculus. The most general version of this property leads to the general integral, or *indefinite integral*, of x^2:

$\frac{1}{3}x^3 + C$ where C is an arbitrary constant;

the indefinite integral is written as $\int f(x)\,\mathrm{d}x$.

The construction of a primitive is, of course, fundamental to the evaluation of the definite integral. The quantity

$$\int_a^b f(x)\,\mathrm{d}x$$

(which, as we have already mentioned, can be interpreted as a specific area) is defined by

$$A(b) - A(a) \text{ where } \frac{\mathrm{d}A}{\mathrm{d}x} = f(x)$$

e.g. the area as far as $x = b$, less the area to $x = a$. In our example above, this becomes

$$\tfrac{1}{3}b^3 + C - \left(\tfrac{1}{3}a^3 + C\right) = \tfrac{1}{3}\left(b^3 - a^3\right),$$

and the arbitrary constant is irrelevant: any primitive will do. (We may observe that we have, in this description of integration, the hierarchy: primitive $\longrightarrow$ indefinite integral $\longrightarrow$ definite integral.)

Example 2

Find the value of $\int_0^1 \mathrm{e}^{3x}\mathrm{d}x$.

A primitive for e^{3x} is $\frac{1}{3}\mathrm{e}^{3x}$ (because $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}\mathrm{e}^{3x}\right) = \mathrm{e}^{3x}$), and so the value of the definite integral is

$$\int_0^1 \mathrm{e}^{3x}\mathrm{d}x = \left[\tfrac{1}{3}\mathrm{e}^{3x}\right]_0^1 = \tfrac{1}{3}\mathrm{e}^3 - \tfrac{1}{3}\mathrm{e}^0 = \tfrac{1}{3}\left(\mathrm{e}^3 - 1\right).$$

1.4 Theorems on integration

The process of integration is often greatly simplified when certain elementary properties (of the underlying differentiation process) are invoked. We will obtain these with the notation

$$\int f(x)\mathrm{d}x = F(x) \text{ is a primitive (so } \frac{\mathrm{d}F}{\mathrm{d}x} = f(x)\text{)}.$$

We assume that $f(x)$ and $F(x)$ both exist throughout the domain that we might wish to use; it is usual to call $f(x)$ the *integrand* (i.e. that which is to be integrated).

Theorem 1

Since we have $\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$, then

$k\frac{\mathrm{d}F}{\mathrm{d}x} = kf(x)$ or $\frac{\mathrm{d}}{\mathrm{d}x}(kF) = kf(x)$ where k is a constant,

thus $\int kf(x)\mathrm{d}x = kF(x) = k\int f(x)\mathrm{d}x$.

Theorem 2

Given $\int_a^b f(x)\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$, ,

then, for $a \le c \le b$,

$$\int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x = [F(x)]_a^c + [F(x)]_c^b$$

$$= F(c) - F(a) + F(b) - F(c) = F(b) - F(a) = \int_a^b f(x)\mathrm{d}x .$$

Thus we have the property

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x, \quad a \le c \le b,$$

and each of these can be further sub-divided (at least a finite number of times).

Theorem 3

Given $\int_a^b f(x)\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$

$$= -[F(a) - F(b)] = -\int_b^a f(x)\mathrm{d}x$$

so ‘reversing the direction of integration’ (b to a rather than a to b) changes the sign of the integral.

Theorem 4

Suppose now that we also have $\int g(x)\mathrm{d}x = G(x)$ where $\dfrac{\mathrm{d}G}{\mathrm{d}x} = g(x)$; we form

$$f(x) + g(x) = \frac{\mathrm{d}F}{\mathrm{d}x} + \frac{\mathrm{d}G}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(F+G),$$

and so we obtain

$$\int \{f(x) + g(x)\}\,\mathrm{d}x = F(x) + G(x)$$

$$= \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x,$$

which describes the *linearity* of the 'integral operator'.

Theorem 5

We are given, now, that $x = h(u)$ (and we assume that this is a one-to-one mapping for all x of interest); we form $F(x) = F[h(u)]$, and then

$$\frac{\mathrm{d}F}{\mathrm{d}x} = f(x) \text{ becomes } \frac{\mathrm{d}}{\mathrm{d}x}F[h(u)] = f[h(u)] \text{ or } \frac{\mathrm{d}F}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = f[h(u)],$$

according to the familiar 'chain rule'. Thus

$$\frac{\mathrm{d}F}{\mathrm{d}u} = f[h(u)]\frac{\mathrm{d}x}{\mathrm{d}u} = f[h(u)]h'(u)$$

(on invoking another familiar rule: $1/(\mathrm{d}u/\mathrm{d}x) = \mathrm{d}x/\mathrm{d}u$, and using the prime as the shorthand for the derivative), and hence

$$F[h(u)] = \int f[h(u)]\,h'(u)\,\mathrm{d}u;$$

but $F[h(u)] = F(x) = \int f(x)\mathrm{d}x$, so we obtain

$$\int f(x)\mathrm{d}x = \int f[h(u)]h'(u)\,\mathrm{d}u$$

for any (suitable) $x = h(u)$.

This result provides the basis for the method of substitution, which is used to find the integral of a function that – apparently – cannot be integrated directly. Indeed, this can be used in a simple form to obtain a particularly illuminating result; given

$$\int f(x)\mathrm{d}x = F(x) \text{ (so that } \frac{\mathrm{d}F}{\mathrm{d}x} = f(x)\text{)},$$

consider the integral

$$\int f(ax+b)\mathrm{d}x$$

where a $(\neq 0)$ and b are real constants. According to Theorem 5, we may introduce $ax+b=u$ (equivalently $x=(u-b)/a \equiv h(u)$), so that $h'(u)=a^{-1}$, which gives

$$\begin{aligned}\int f(ax+b)\mathrm{d}x &= \int f(u)\frac{1}{a}\mathrm{d}u \\ &= \frac{1}{a}\int f(u)\mathrm{d}u = \frac{1}{a}F(u) \\ &= \frac{1}{a}F(ax+b)\,.\end{aligned}$$

[Check: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{a}F(ax+b)\right] = F'(ax+b) = f(ax+b)$.]

In other words, whenever we are able to integrate $f(x)$, then we can always integrate $f(ax+b)$.

Another important application of this property arises when the integrand, as given, has the structure of the derivative of 'a function of a function'. Consider

$$\int f[g(x)]g'(x)\,\mathrm{d}x \text{ and set } u = g(x) \text{ i.e. } \frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$$

(which, as described above, we assume is a one-to-one function, so that x is defined by $x = g^{-1}(u)$ and is unique); thus we obtain

$$\int f(u)\,\mathrm{d}u = F(u)$$

if we suppose that $F(u)$ is a primitive of $\int f(u)\,\mathrm{d}u$.

1.5 Standard integrals

The definition

$$\int f(x)\mathrm{d}x = F(x) \text{ (i.e. } \mathrm{d}F/\mathrm{d}x = f(x))$$

of a primitive enables us to produce a list of common primitives, based on elementary differentiation. These, coupled with the various properties described in §1.4, enable a number of integration problems to be solved directly. Although this is useful, and it is an essential starting point, this barely represents what is generally regarded as the mathematical exercise (and challenge) that is 'integration'. It is this latter aspect that will be the main thrust of Part I. Nevertheless, we should record the most common primitives that are met in elementary mathematics; these are provided in the list below.

$f(x)$	$F(x) = \int f(x)\mathrm{d}x$
x^n (any $n \neq -1$)	$x^{n+1}/(n+1)$
x^{-1}	$\ln\|x\|$
e^{ax}	e^{ax}/a
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\operatorname{cosec}^2 x$	$-\cot x$
$1/(x^2+a^2)$	$(1/a)\arctan(x/a)$
$1/\sqrt{a^2-x^2}$	$\arcsin(x/a)$ (or $-\arccos(x/a)$)
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\operatorname{sech}^2 x$	$\tanh x$
$1/\sqrt{x^2+a^2}$	$\operatorname{arcsinh}(x/a)$ (or $\ln\left\|x+\sqrt{x^2+a^2}\right\|$)
$1/\sqrt{x^2-a^2}$	$\operatorname{arccosh}(x/a)$ (or $\ln\left\|x+\sqrt{x^2-a^2}\right\|$)
$\tan x$	$-\ln\|\cos x\|$
$\cot x$	$\ln\|\sin x\|$

Comment: Most of these should be familiar to the reader but, wherever there is any doubt, simply find $\mathrm{d}F/\mathrm{d}x$ to check the result in the table. The one entry which, although generally accepted, sometimes causes unease is $f(x) = x^{-1}$, $F(x) = \ln|x|$; we briefly explain this. Suppose, first, that $x > 0$, then

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x};\ ;$$

on the other hand, if we have $x = -y < 0$ (so that $y > 0$), then again

$$\frac{\mathrm{d}}{\mathrm{d}y}(\ln y) = \frac{1}{y}\cdot$$

But this can be expressed as

$$\frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(-x)\right] = -\frac{1}{x},$$

where $\mathrm{d}x/\mathrm{d}y = -1$ i.e. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln(-x)\right] = \dfrac{1}{x}$ for $x < 0$;

thus we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln|x| = \frac{1}{x} \quad \text{or} \quad \int \frac{1}{x}\,\mathrm{d}x = \ln|x| \quad \text{for all } x \neq 0$$

Example 3

Write down primitives associated with the integrals of each of these functions:

$$\text{(a) } (2x+1)^5\text{; (b) } (3-4x)^{3/2}\text{; (c) } (3x+4)^{-1}\text{; (d) } 2\sin(17x-6)\text{; (e) } \tan(1-2x)\text{;}$$
$$\text{(f) } \frac{3}{x^2+16}\text{; (g) } \frac{5}{\sqrt{x^2-2x-8}}\text{; (h) } x^2\sin\left(x^3\right)\text{; (i) } \frac{1+2x}{x^2+x-3}.$$

All these follow directly from the standard results, together with some of the theorems presented in §1.4; so we obtain

(a) $\frac{1}{12}(2x+1)^6$; (b) $-\frac{1}{10}(3-4x)^{5/2}$; (c) $\frac{1}{3}\ln|3x+4|$; (d) $-\frac{2}{17}\cos(17x-6)$;
(e) $\frac{1}{2}\ln|\cos(1-2x)|$; (f) $\frac{3}{4}\arctan(x/4)$;
(g) first write $x^2-2x-8=(x-1)^2-9=(x-1)^2-3^2$, then we obtain
$5\ln\left|x-1+\sqrt{x^2-2x-8}\right|$; (h) $-\frac{1}{3}\cos(x^3)$ (since we may set $u=x^3$);
(i) $\ln\left|x^2+x-3\right|$ (since we may set $u=x^2+x-3$).

As a check, you should differentiate these answers in order to confirm that $\mathrm{d}F/\mathrm{d}x = f(x)$.

Now let us work through a few definite integrals.

Example 4

Evaluate these definite integrals:

$$\text{(a) } \int_1^2 \left[x^2+(3x-1)^{-2}\right]\mathrm{d}x\text{ ; (b) } \int_0^{1/4} 3\sec^2\left(\frac{\pi}{4}-x\pi\right)\mathrm{d}x\text{ ; (c) } \int_1^2 \frac{\mathrm{d}x}{\sqrt{9-x^2}}$$

To answer these, we first find a primitive.

$$\text{(a) } \int_1^2 \left[x^2+(3x-1)^{-2}\right]\mathrm{d}x = \left[\tfrac{1}{3}x^3-\tfrac{1}{3}(3x-1)^{-1}\right]_1^2 = \tfrac{8}{3}-\tfrac{1}{3}.\tfrac{1}{5}-\left(\tfrac{1}{3}-\tfrac{1}{3}.\tfrac{1}{2}\right)$$

$$= \tfrac{7}{3}+\tfrac{1}{3}\left(\tfrac{1}{2}-\tfrac{1}{5}\right) = \tfrac{7}{3}+\tfrac{1}{10} = \frac{73}{30};$$

$$\text{(b) } \int_0^{1/4} 3\sec^2\left(\frac{\pi}{4}-x\pi\right)\mathrm{d}x = \left[-\frac{3}{\pi}\tan\left(\frac{\pi}{4}-x\pi\right)\right]_0^{1/4} = -\frac{3}{\pi}\tan 0-\left(-\frac{3}{\pi}\tan\left(\frac{\pi}{4}\right)\right) = \frac{3}{\pi};$$

$$\text{(c) } \int_1^2 \frac{\mathrm{d}x}{\sqrt{9-x^2}} = \left[\arcsin(x/3)\right]_1^2 = \arcsin(2/3)-\arcsin(1/3).$$

1.6 Integration by parts

This important and powerful result is obtained from the standard rule for the derivative of a product:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uw) = u'w + uw'$$

(where the prime denotes the derivative), so that

$$\begin{aligned} uw &= \int (wu' + uw')\,\mathrm{d}x \\ &= \int wu'\,\mathrm{d}x + \int uw'\,\mathrm{d}x \quad \text{(Theorem 4).} \end{aligned}$$

Now let w be any primitive of v i.e. $w = \int v\,\mathrm{d}x$, so that $w' = v$, then we may write

$$u\int v\,\mathrm{d}x = \int u'\left(\int v\,\mathrm{d}x\right)\mathrm{d}x + \int uv\,\mathrm{d}x$$

or
$$\int uv\,\mathrm{d}x = u\left(\int v\,\mathrm{d}x\right) - \int u'\left(\int v\,\mathrm{d}x\right)\mathrm{d}x$$

which is the required formula. In the case of a definite integral, we have

$$[uw]_a^b = \int_a^b wu'\,\mathrm{d}x + \int_a^b uw'\,\mathrm{d}x$$

and so $\displaystyle\int_a^b uv\,\mathrm{d}x = \left[u\int v\,\mathrm{d}x\right]_a^b - \int_a^b u'\left(\int v\,\mathrm{d}x\right)\mathrm{d}x$ . .

Note that, in each formula, the term $\int v\,\mathrm{d}x$ is any primitive of $v(x)$; further observe that, in the case of the definite integral, the product '$u\int v\,\mathrm{d}x$' is formed before any evaluation is imposed.

Example 5

Use the method of integration by parts to find these definite and indefinite integrals:

(a) $\int x^n \ln x\,\mathrm{d}x$ ($x > 0$, $n \neq -1$); (b) $\int \arctan x\,\mathrm{d}x$; (c) $\int_0^1 x^2 \mathrm{e}^{-x}\mathrm{d}x$; (d) $\int_0^2 \sqrt{4-x^2}\,\mathrm{d}x$.

(a) We choose to differentiate $\ln x$ and to integrate x^n in the integrand, and so

$$\int x^n \ln x\,dx = (\ln x)\left(\frac{x^{n+1}}{n+1}\right) - \int\left(\frac{x^{n+1}}{n+1}\right)\left(\frac{1}{x}\right)dx$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n\,dx$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C = \left(\ln x - \frac{1}{n+1}\right)\left(\frac{x^{n+1}}{n+1}\right) + C,$$

where C is the arbitrary constant of integration.

(b) In this case we – apparently – do not have a product, but we may always generate one by using the interpretation: $\arctan x = 1 \times \arctan x$ and then (obviously) we would elect to integrate the '1', so

$$\int \arctan x\,dx = \int 1 \times \arctan x\,dx = x\arctan x - \int x\left(\frac{1}{1+x^2}\right)dx$$
$$= x\arctan x - \tfrac{1}{2}\ln\left(1+x^2\right) + C,$$

where C is the arbitrary constant of integration.

(c) This problem requires two integrations by parts:

$$\int_0^1 x^2 e^{-x} dx = \left[-e^{-x}x^2\right]_0^1 - \int_0^1 \left(-e^{-x}\right).2x\,dx$$

$$= -e^{-1} + 2\left\{\left[-e^{-x}x\right]_0^1 - \int_0^1 \left(-e^{-x}\right).1\,dx\right\}$$

$$= -e^{-1} + 2\left\{-e^{-1} + \left[-e^{-x}\right]_0^1\right\} = 2 - 5e^{-1}$$

(and you might wish to check that $2 - 5e^{-1} > 0$, for it is clear that $\int_0^1 x^2 e^{-x} dx > 0$)).

(d) We follow the same route as used in (b): write $\sqrt{4-x^2} = 1 \times \sqrt{4-x^2}$, to give

$$\int_0^2 1 \times \sqrt{4-x^2}\,dx = \left[x\sqrt{4-x^2}\right]_0^2 - \int_0^2 x.\frac{-x}{\sqrt{4-x^2}}\,dx$$

$$= \int_0^2 \frac{x^2}{\sqrt{4-x^2}}\,dx = -\int_0^2 \frac{4-x^2-4}{\sqrt{4-x^2}}\,dx$$

$$= -\int_0^2 \left\{\sqrt{4-x^2} - \frac{4}{\sqrt{4-x^2}}\right\} dx = -\int_0^2 \sqrt{4-x^2}\,dx + 4\int_0^2 \frac{dx}{\sqrt{4-x^2}}.$$

This result does not look promising, until we observe that we have generated an *equation* for the required integral:

$$2\int_0^2 \sqrt{4-x^2}\,dx = 4\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \left[4\arcsin\left(\frac{x}{2}\right)\right]_0^2 = 4.\frac{\pi}{2}$$

$$\int_0^2 \sqrt{4-x^2}\,dx = \pi.$$

Comment: This result could have been obtained without integration; the integral represents the area under the curve $y = \sqrt{4-x^2}$, $0 \le x \le 2$, which is one quarter of the area of the circle of radius 2, so the required area is $\frac{1}{4} \times \pi(2)^2 = \pi$.

The technique employed in (b) and (d) above is an important one, and it can be generalised to the extent that we may write

$$\int f(x)\,\mathrm{d}x = \int \frac{g(x)f(x)}{g(x)}\,\mathrm{d}x\,,$$

for suitable $g(x)$, if that is useful.

Example 6

Use integration by parts to find an equation involving $\int (1+\ln x)^n\,\mathrm{d}x$ ($x>0$, $n \neq -1$); hence find the value of $\int_1^2 (1+\ln x)^3\,\mathrm{d}x$.

In this case, we note that it would be very convenient if we had $\frac{1}{x}(1+\ln x)^n$ (because $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$), so let us write

$$\int (1+\ln x)^n\,\mathrm{d}x = \int x.\frac{1}{x}.(1+\ln x)^n\,\mathrm{d}x$$

$$= x.\frac{(1+\ln x)^{n+1}}{n+1} - \int \frac{(1+\ln x)^{n+1}}{n+1}.1\,\mathrm{d}x\,.$$

Now, at first sight, this appears not to be helpful, but when we set $I_n = \int (1+\ln x)^n\,\mathrm{d}x$, our expression above becomes

$$I_n = \frac{x}{n+1}(1+\ln x)^{n+1} - \frac{1}{n+1} I_{n+1}$$

which is a *reduction formula* (or *recurrence relation*). We have, alternatively,

$$I_n = x(1+\ln x)^n - nI_{n-1}$$

where we have written *n* for *n* + 1. The definite-integral version of this is

$$J_n = \left[x(1+\ln x)^n\right]_1^2 - nJ_{n-1} \quad \left(J_n = \int_1^2 (1+\ln x)^n\,\mathrm{d}x\right)$$

$$= 2(1+\ln 2)^n - 1 - nJ_{n-1}\,,$$

and so

$$\begin{aligned} J_3 &= 2(1+\ln 2)^3 - 1 - 3J_2 \\ &= 2(1+\ln 2)^3 - 1 - 3\left\{2(1+\ln 2)^2 - 1 - 2J_1\right\} \\ &= 2(1+\ln 2)^3 - 6(1+\ln 2)^2 + 2 + 6\left\{2(1+\ln 2) - 1 - J_0\right\} \end{aligned}$$

where . $J_0 = \int_1^2 1.\mathrm{d}x = [x]_1^2 = 1$. Thus we have

$$\int_1^2 (1+\ln x)^3 \mathrm{d}x = 2(1+\ln 2)\left[1 + 2\ln 2 + (\ln 2)^2 - 3 - 3\ln 2 + 6\right] - 10$$

$$= 2(\ln 2)^3 + 6\ln 2 - 2 .$$

The idea used in this example is often employed in order to find a representation of integrals (or the value of definite integrals) that have a structure that suggests a pattern (or is one of a pattern that might be proposed). Thus, in the case of , $\int_1^2 (1+\ln x)^3 \mathrm{d}x$, we might first consider , $\int_1^2 (1+\ln x)^n \mathrm{d}x$, obtain a reduction formula and then proceed to find the value that we require – which, of course, is exactly the route that we followed in Example 6. This is one of the most powerful applications of the method of integration by parts; we will therefore present another example of this type.

Example 7

Find a reduction formula for $I_n = \int \tan^n \theta \mathrm{d}\theta$ ($n \geq 2$) and hence construct a primitive for $\int \tan^6 \theta \mathrm{d}\theta$. .

We write $\int \tan^n \theta \mathrm{d}\theta = \int \tan^2 \theta . \tan^{n-2} \theta \mathrm{d}\theta$

$$= \int \left(\sec^2 \theta - 1\right) \tan^{n-2} \theta \mathrm{d}\theta$$

$$= \frac{1}{n-1} \tan^{n-1} \theta - I_{n-2}$$

which is a reduction formula. Now we can use this as follows:

$$I_6 = \tfrac{1}{5} \tan^5 \theta - I_4 = \tfrac{1}{5} \tan^5 \theta - \left(\tfrac{1}{3} \tan^3 \theta - I_2\right)$$

$$= \tfrac{1}{5} \tan^5 \theta - \tfrac{1}{3} \tan^3 \theta + \tan \theta - I_0$$

where $I_0 = \int 1.\mathrm{d}\theta = \theta$ is a primitive; hence a primitive for $\int \tan^6 \theta \mathrm{d}\theta$ is

$$\tfrac{1}{5} \tan^5 \theta - \tfrac{1}{3} \tan^3 \theta + \tan \theta - \theta$$

(which can be checked by differentiation).

The process of integration involves one further (minor) complication that we must address, although – as we shall see – this does lead to an important general comment about integration and the notion of the existence of integrals.

1.7 Improper integrals

In all our previous considerations, we have assumed that the function to be integrated – the integrand – is bounded and, further, that the integration region (from *a* to *b*, say) is a finite domain. When either or both these requirements are not met, then we have an *improper integral*; we shall now examine these two possibilities.

We consider the definite integral

$$\int_a^b f(x)\,\mathrm{d}x \quad \text{with primitive } F(x);$$

let us suppose that *a* is fixed, but that $b \to \infty$. (We can equally consider *b* fixed and allow $a \to -\infty$, or even $b \to \infty$ and $a \to -\infty$, as simple extensions of the argument that we are about to present.) From the definition of a definite integral we have

$$\int_a^b f(x)\,\mathrm{d}x = \left[F(x)\right]_a^b = F(b) - F(a)\,;$$

;

if $F(b) - F(a)$ has a finite limit as $b \to \infty$, then the integral exists i.e. it is uniquely defined and finite. (It is 'improper' because of the infinite limit of integration, and this interpretation of the value of the definite integral becomes necessary.) Thus we define, when it exists,

$$\int_a^\infty f(x)\,\mathrm{d}x \;\text{ as }\; \lim_{b\to\infty}\left(\int_a^b f(x)\,\mathrm{d}x\right).$$

Example 8

Decide whether these integrals exist; when an integral exists, find its value.

$$\text{(a) } \int_0^\infty \frac{\mathrm{d}x}{1+x^2}\,;\ \text{(b) } \int_0^\infty \sin x\,\mathrm{d}x\,;\ \text{(c) } \int_0^\infty \frac{\mathrm{d}x}{1+x}\,.$$

In each case we form $\int_0^b f(x)\,\mathrm{d}x = \left[F(x)\right]_0^b = F(b) - F(0)$ and then allow $b \to \infty$:

(a) $\displaystyle\int_0^b \frac{\mathrm{d}x}{1+x^2} = \left[\arctan x\right]_0^b = \arctan b \to \frac{\pi}{2}$ as $b \to \infty$, so the integral exists and takes

this value;

(b) $\displaystyle\int_0^b \sin x\,\mathrm{d}x = \left[-\cos x\right]_0^b = 1 - \cos b$, but this does not have a limit as $b \to \infty$ (although

it does remain bounded, it forever oscillates between 0 and 2): the integral does not

exist;

(c) $\displaystyle\int_0^b \frac{\mathrm{d}x}{1+x} = \left[\ln(1+x)\right]_0^b = \ln b - \ln 1 = \ln b$, but $\ln b \to \infty$ as $b \to \infty$, so this integral

does not exist.

An integral is also called 'improper' if $f(x)$ is undefined at any point over the range of integration e.g. $\int_{-1}^{1} \mathrm{d}x/x$ is improper because the function $f(x) = x^{-1}$ is undefined at $x = 0$ which is in the range of integration. However, the integrals of such functions may exist. Consider

$$\int_{a+\varepsilon}^{b} f(x)\,\mathrm{d}x \qquad \left(\text{with } \frac{\mathrm{d}F}{\mathrm{d}x} = f(x)\right),$$

where $\varepsilon > 0$ and $f(x)$ exists for $x \in [a+\varepsilon, b]$ but is undefined at $x = a$. (Any integral with one – or more – values of x at which $f(x)$ does not exist can be written as a sum of integrals – Theorem 2 – to produce a contribution as described above or, equivalently, one like $\int_a^{b-\varepsilon} f(x)\,\mathrm{d}x$, or a combination of the two.) Thus we have

$$\int_{a+\varepsilon}^{b} f(x)\,\mathrm{d}x = [F(x)]_{a+\varepsilon}^{b} = F(b) - F(a+\varepsilon)$$

and if $F(b) - F(a+\varepsilon)$ has a limit as $\varepsilon \to 0$, then the integral exists. Thus we define, when it exists,

$$\int_{a}^{b} f(x)\,\mathrm{d}x \ \text{ as } \lim_{\varepsilon \to 0^+}\left(\int_{a+\varepsilon}^{b} f(x)\,\mathrm{d}x\right),$$

where $\varepsilon \to 0^+$ denotes an approach to zero through the positive numbers.

Example 9

Decide whether these integrals exist; when an integral exists, find its value.

$$\int_0^1 \frac{dx}{\sqrt{x}}\text{;(b) } \int_{-1}^2 \frac{dx}{1+x}\text{; (c) } \int_0^1 \ln x \, dx \text{ .}$$

(a) $\int_\varepsilon^1 \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_\varepsilon^1 = 2 - 2\sqrt{\varepsilon} \to 2$ as $\varepsilon \to 0^+$, so the integral exists and has this value;

(b) $\int_{-1+\varepsilon}^2 \frac{dx}{1+x} = [\ln(1+x)]_{-1+\varepsilon}^2 = \ln 3 - \ln \varepsilon \to +\infty$ as $\varepsilon \to 0^+$, so the integral does not exist;

(c) $\int_\varepsilon^1 \ln x \, dx = [x \ln x - x]_\varepsilon^1 = -1 - (\varepsilon \ln \varepsilon - \varepsilon) \to -1$ as $\varepsilon \to 0^+$ as $\varepsilon \to 0^+$ (because $\varepsilon \ln \varepsilon \to 0^-$ as $\varepsilon \to 0^+$), so this integral exists and has this value.

1.8 Non-uniqueness of representation

We finally comment on one other aspect of integration which, at first sight, may appear irritating but which is ultimately a powerful tool. We have seen that the process of integration is not unique in that we can produce any number of primitives, each differing from the other by additive constants. However, there often exists a non-uniqueness of representation i.e. different methods of integration may give results that appear different (and then, of course, in a particular context we may prefer one rather than another). A simple example appears in the table of elementary integrals (p. 15):

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x \;\text{ or }\; -\arccos x \text{,}$$

yet this non-unique representation has important consequences.

We have

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} = \frac{d}{dx}(-\arccos x)$$

and so we obtain

$$\frac{d}{dx}(\arcsin x + \arccos x) = 0$$

i.e. $\arcsin x + \arccos x = \text{constant}$.

Evaluation on, for example, $x = 0$ yields

$$\arcsin x + \arccos x = \frac{\pi}{2},$$

a fundamental identity.

Example 10

Confirm that $\arctan(\sinh x)$, $2\arctan(\mathrm{e}^x)$, $2\arctan(\tanh(x/2))$ and $\arcsin(\tanh x)$ are all primitives of $\int \operatorname{sech} x \,\mathrm{d}x$.

For each, we find the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\arctan(\sinh x)\right] = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x\,;$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[2\arctan(\mathrm{e}^x)\right] = \frac{2\mathrm{e}^x}{1+\mathrm{e}^{2x}} = \frac{1}{\frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)} = \operatorname{sech} x\,;$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[2\arctan(\tanh(x/2))\right] = \frac{2.\frac{1}{2}\operatorname{sech}^2(x/2)}{1+\tanh^2(x/2)} = \frac{1}{\sinh^2(x/2)+\cosh^2(x/2)}$$
$$= \frac{1}{\cosh x} = \operatorname{sech} x\,;$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\arcsin(\tanh x)\right] = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\operatorname{sech} x} = \operatorname{sech} x\,.$$

Exercises 1

1. Find all these definite and indefinite integrals (when they exist):

$$\text{(a) } \int (3-5x)^9\,\mathrm{d}x\,;\ \text{(b) } \int \tan(2x-1)\,\mathrm{d}x\,;\ \text{(c) } \int_0^1 \frac{\mathrm{d}x}{\sqrt{x^2+2x+2}}\,;\ \text{(d) } \int x^2 \sin x\,\mathrm{d}x\,;$$

$$\text{(e) } \int_0^1 \arcsin x\,\mathrm{d}x\,;\ \text{(f) } \int_1^\infty \frac{\ln x}{x^2}\,\mathrm{d}x\,;\ \text{(g) } \int_1^\infty \frac{\ln x}{x}\,\mathrm{d}x\,;\ \text{(h) } \int x\exp\left(x^2\right)\mathrm{d}x\,;$$

$$\text{(i) } \int_1^2 \frac{3x^2-4x+1}{x^3-2x^2+x+1}\,\mathrm{d}x\,.$$

2. Find reduction formulae for

$$\text{(a)}\ I_n = \int x^{\alpha}(\ln x)^n \, dx \ \ (n \geq 1,\ \alpha \ \text{fixed}); \ \ \text{(b)}\ I_n = \int \sec^n x \, dx \ \ (n \geq 2)$$

3. Integrate $\int \frac{dx}{1+x^2}$, according to the list of integrals on p.15; now use the substitution $x = \frac{1-\alpha \tan u}{\alpha + \tan u}$ (where α is a constant), integrate and hence deduce the standard identity

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

2 The integration of rational functions

It is quite common to encounter functions that are *rational*, that is, they take the form of (polynomial)/(polynomial). Indeed, a recognised and important skill in basic mathematics is knowing how to rewrite such functions so as to make the process of integration almost routine. The aim, in this chapter, is to develop a systematic approach to these problems, at least for a reasonable class of polynomials (and certainly the ones most often encountered). Let us first itemise the polynomials that we shall discuss.

We will investigate

$$\int f(x)\,\mathrm{d}x \text{ with } f(x)=\frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials. We allow $P(x)$ to be a polynomial of any (finite) degree, but we shall restrict the choices for $Q(x)$ (although what we present should make it possible to extend the ideas to more complicated *Q*s). (Remember that the degree of a polynomial is the highest power of *x* that appears.) The simplest $Q(x)$ is $x+a$, where $a \neq 0$ is a constant; the case $a=0$ produces a trivial problem. Note that the coefficient of *x* has been absorbed into $P(x)$. A simple extension of this choice for *Q* is $Q(x)=(x+a)^m$ ($m>1$, integer), which we shall also discuss. Then we consider $Q(x)=x^2+2bx+c$ (where the constant *c* is not zero); this problem gives rise to three cases, depending on whether $x^2+2bx+c=0$ has two distinct, real roots, a repeated root or a complex pair of roots. Finally, we shall examine in detail the choice $Q(x)=(x+a)\left(x^2+2bx+c\right)$, although we will take the opportunity to comment briefly on some of the obvious generalisations and extensions of all these.

2.1 Improper fractions

The first stage in the simplification process is to reduce the rational function so that its fractional part is 'proper' i.e. the degree of the polynomial in the numerator is less than that in the denominator. This is accomplished by dividing – but note below how we accomplish this – $Q(x)$ into $P(x)$. We start with

$$\frac{P(x)}{Q(x)}=\frac{x^n+b_{n-1}x^{n-1}+\ldots+b_0}{x^m+a_{m-1}x^{m-1}+\ldots+a_0}$$

where $n \geq m$ (which defines the fraction to be improper); the division then leaves, at worst, a fraction $\hat{P}(x)/Q(x)$ with $\hat{P}(x)$ a polynomial of degree $m-1$: this is a proper fraction. Once this has been completed, the integration can begin.

Example 11

Reduce $\dfrac{x^5-3x^4+5x^3-2x^2+x-1}{x^2+x+1}$ to a form that contains, at worst, a proper fraction.

First we write

$$\begin{aligned} &x^5-3x^4+5x^3-2x^2+x-1 \\ &\quad = x^3\left(x^2+x+1\right)-4x^2\left(x^2+x+1\right)+8x\left(x^2+x+1\right)-6\left(x^2+x+1\right)-x+5 \end{aligned}$$

and then we have

$$\frac{x^5-3x^4+5x^3-2x^2+x-1}{x^2+x+1}=x^3-4x^2+8x-6+\frac{5-x}{x^2+x+1},$$

which is the required form.

2.2 Linear denominator, Q(x)

We consider

$$f(x)=\frac{P(x)}{Q(x)}=\frac{x^n+b_{n-1}x^{n-1}+....+b_0}{x+a}$$

which can be written in the form

$$f(x)=x^{n-1}+c_{n-2}x^{n-2}+....+c_0+\frac{d}{x+a}$$

for appropriate coefficients c_0, c_1, …, c_{n-2} and *d*. It is immediately apparent that each of these terms can be integrated – they are all elementary functions – so we can always find $\int f(x)\,\mathrm{d}x$ in this case.

Example 12

Find $\int \frac{x^3-x^2+2x+1}{x+1}\,\mathrm{d}x$.

First we write $x^3-x^2+2x+1=x^2(x+1)-2x(x+1)+4(x+1)-3$, and then the integral becomes

$$\int\left(x^2-2x+4-\frac{3}{x+1}\right)\mathrm{d}x=\tfrac{1}{3}x^3-x^2+4x-3\ln|x+1|+C,$$

where *C* is the arbitrary constant of integration.

Comment: It should be clear that we can generalise this result to the situation where $Q(x)=(x+a)^m$ ($m>1$, integer). In this case, the fractions that appear, in general, will take the form

$$\frac{d_1}{x+a},\frac{d_2}{(x+a)^2},....,\frac{d_m}{(x+a)^m},$$

and each of these can be integrated.

Example 13

Find $\int \frac{x^6 - 3x^5 + 2x^4 + x^3 - x^2 + 4x - 7}{(x-2)^3} \mathrm{d}x$.

First we write the numerator as

$$x^5(x-2) - x^4(x-2) + x^2(x-2) + x(x-2) + 6(x-2) + 5$$

$$= (x-2)\left[x^4(x-2) + x^3(x-2) + 2x^2(x-2) + 5x(x-2) + 11(x-2) + 28\right] + 5$$

$$= (x-2)^2\left[x^3(x-2) + 3x^2(x-2) + 8x(x-2) + 21(x-2) + 53\right] + 28(x-2) + 5.$$

Thus the integral becomes

$$\int \left\{ x^3 + 3x^2 + 8x + 21 + \frac{53}{x-2} + \frac{28}{(x-2)^2} + \frac{5}{(x-2)^3} \right\} \mathrm{d}x$$

$$= \tfrac{1}{4}x^4 + x^3 + 4x^2 + 21x + 53\ln|x-2| - \frac{28}{(x-2)} - \frac{5}{2(x-2)^2} + C .$$

2.3 Quadratic denominator, Q(x)

This case is significantly more involved than the previous one, essentially because there are three variants (depending on the nature of the roots of $Q(x)=0$) each leading to a different integration procedure. Let us be given

$$f(x)=\frac{P(x)}{Q(x)}=\frac{x^n+a_{n-1}x^{n-1}+\ldots+a_0}{x^2+2bx+c};$$

further, we will suppose that this improper fraction is reduced to a proper fraction (and a suitable polynomial), as described earlier. Thus we need to consider in detail only the term

$$\frac{px+q}{x^2+2bx+c},$$

where p and q are constants. To proceed, we write

$$x^2+2bx+c=(x+b)^2+c-b^2$$

and then we consider the three cases:

$$c-b^2<0,\ c-b^2=0,\ c-b^2>0.$$

Case (1): $c-b^2<0$

It is convenient to write $c-b^2=-\lambda^2(<0)$, so that

$$x^2+2bx+c=(x+b)^2-\lambda^2=(x+b-\lambda)(x+b+\lambda)$$

and then it is simply a matter of expressing

$$\frac{px+q}{(x+b-\lambda)(x+b+\lambda)}$$

in partial fractions; integration then follows directly.

Example 14

Find $\int\frac{x^3-1}{x^2+4x+3}\mathrm{d}x$.

We first write $x^3-1=x(x^2+4x+3)-4(x^2+4x+3)+13x+11$

so that the integral becomes

$$\int\left(x-4+\frac{13x+11}{x^2+4x+3}\right)\mathrm{d}x.$$

But $x^2 + 4x + 3 = (x+2)^2 - 1 = (x+1)(x+3)$ (which you probably realised immediately), and so we write

$$\frac{13x+11}{x^2+4x+3} = \frac{13x+11}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$$

which requires $A = -1$ and $B = 14$, and this produces the integral in the form

$$\int \left(x - 4 + \frac{14}{x+3} - \frac{1}{x+1} \right) \mathrm{d}x = \tfrac{1}{2}x^2 - 4x + 14\ln|x+3| - \ln|x+1| + C .$$

Case (2): $c - b^2 = 0$

This is particularly straightforward: $Q(x) = (x+b)^2$, so we have the case $m = 2$ in the generalisation $Q(x) = (x+a)^m$ included in §2.2.

Example 15

Find the value of $\int_1^2 \frac{x^4 - 2x^2 + 3x - 2}{x^2 - 6x + 9} \mathrm{d}x$.

First we note that $x^2 - 6x + 9 = (x-3)^2$, and so we write

$$x^4 - 2x^2 + 3x - 2 = x^3(x-3) + 3x^2(x-3) + 7x(x-3) + 24(x-3) + 70$$

$$= (x-3)\left[x^2(x-3) + 6x(x-3) + 25(x-3) + 99\right] + 70 ;$$

the integral therefore becomes

$$\int_1^2 \left\{ x^2 + 6x + 25 + \frac{99}{x-3} + \frac{70}{(x-3)^2} \right\} \mathrm{d}x$$

$$= \left[\tfrac{1}{3}x^3 + 3x^2 + 25x + 99\ln|x-3| - \frac{70}{(x-3)} \right]_1^2$$

$$= \tfrac{8}{3} + 12 + 50 + 70 + 99\ln 1 - \left(\tfrac{1}{3} + 3 + 25 + 99\ln 2 + 353 \right) = \frac{214}{3} - 99\ln 2 .$$

Case (3): $c - b^2 > 0$

Now we set $c - b^2 = \lambda^2 (> 0)$ and then $(x+b)^2 + \lambda^2$ cannot be factorised (into real factors). The division $P(x)/Q(x)$ will therefore produce, in general, the fraction

$$\frac{px+q}{(x+b)^2+\lambda^2}$$

which is rewritten, first, to accommodate the derivative of the denominator:

$$\frac{\frac{1}{2}p\{2(x+b)\}+q-pb}{(x+b)^2+\lambda^2}=\frac{p}{2}\frac{2(x+b)}{(x+b)^2+\lambda^2}+\frac{q-pb}{(x+b)^2+\lambda^2}.$$

These two terms can be integrated, the first producing a logarithmic term and the second (after a substitution) is an *arctan* term. Thus we are able to complete the integration. (Note that there is no suggestion that the 'formula' just derived should be committed to memory – it is simply presented to make clear that the problem can be solved.)

Example 16

Find $\int \frac{x^3-3x+1}{x^2-8x+25}\mathrm{d}x$.

We observe that $x^2-8x+25=(x-4)^2+9$, and then we write

$$x^3-3x+1=x(x^2-8x+25)+8(x^2-8x+25)+36x-199$$

to give the integral in the form

$$\int\left(x+8+\frac{36x-199}{x^2-8x+25}\right)\mathrm{d}x$$
$$=\int\left(x+8+\frac{18(2x-8)-55}{x^2-8x+25}\right)\mathrm{d}x$$
$$=\tfrac{1}{2}x^2+8x+18\ln\left(x^2-8x+25\right)-\int\frac{55}{(x-4)^2+9}\mathrm{d}x\,.$$

In the last term, we set $x-4=3\tan u$ to give $55\int\frac{3\sec^2 u}{9\left(1+\tan^2 u\right)}\mathrm{d}u=\frac{55}{3}\int\mathrm{d}u=\frac{55}{3}u$ (a primitive); hence the required integral is

$$\tfrac{1}{2}x^2+8x+18\ln\left(x^2-8x+25\right)-\tfrac{55}{3}\arctan\left(\frac{x-4}{3}\right)+C\,.$$

2.4 Cubic denominator, Q(x)

We start here with $Q(x)=x^3+\alpha x^2+\beta x+\gamma$, and recall that a cubic equation [$Q(x)=0$] with real coefficients must have at least one real root; let this be $x=-a$, then we shall write

$$Q(x)=(x+a)(x^2+2bx+c)\,.$$

Thus our integrand becomes

$$f(x)=\frac{P(x)}{Q(x)}=\frac{x^n+d_{n-1}x^{n-1}+\ldots.+d_0}{(x+a)(x^2+2bx+c)}$$

which will, after division by $Q(x)$, reduce to the problem of integrating (in general)

$$\frac{rx^2+px+q}{(x+a)(x^2+2bx+c)}\,.$$

This, in turn, can be expressed as the two partial fractions

$$\frac{A}{x+a}+\frac{Bx+C}{x^2+2bx+c}\,,$$

which can be further developed, depending on the value of $c-b^2$ (exactly as in §2.3): the integration can be completed.

Example 17

Find $\displaystyle\int\frac{x^6-x^3+1}{x^3+x^2+x+1}\mathrm{d}x\cdot$

First we note that $x^3+x^2+x+1=(x+1)(x^2+1)$, so we have two (real) factors only; then we write

$$x^6-x^3+1=x^3(x^3+x^2+x+1)-x^2(x^3+x^2+x+1)$$

$$-(x^3+x^2+x+1)+2x^2+x+2\,.$$

Thus we have the integral

$$\int\left(x^3-x^2-1+\frac{2x^2+x+2}{x^3+x^2+x+1}\right)\mathrm{d}x$$

where we may express

$$\frac{2x^2+x+2}{x^3+x^2+x+1}=\frac{2x^2+x+2}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

which requires $A=3/2$, $B=C=1/2$, so now our integral becomes

$$\int\left(x^3-x^2-1+\frac{3/2}{x+1}+\frac{1}{2}\frac{\frac{1}{2}(2x)+1}{x^2+1}\right)\mathrm{d}x$$

$$=\tfrac{1}{4}x^4-\tfrac{1}{3}x^3-x+\tfrac{3}{2}\ln|x+1|+\tfrac{1}{4}\ln\left(x^2+1\right)+\tfrac{1}{2}\arctan x+C\,.$$

Comment: Some generalisations of these techniques are fairly straightforward. For example, a polynomial $Q(x)$ that possesses any number of linear factors (including repeated factors), and any number of quadratic factors (but none repeated), can be integrated by applying the techniques just outlined. Here is an example of this approach, but one in which some of the quadratic factors cannot be further reduced to linear factors.

Example 18

Find $\displaystyle\int\frac{x-2}{(x+1)(x^2-1)(x^2+2x+2)(x^2+2x+5)}\mathrm{d}x$..

Because the degree of the numerator is already less than that in the denominator, we need simply to represent the integrand as suitable partial fractions. Thus we write

$$\frac{x-2}{(x+1)(x^2-1)(x^2+2x+2)(x^2+2x+5)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}$$

$$+\frac{Dx+E}{x^2+2x+2}+\frac{Fx+G}{x^2+2x+5}$$

which requires $A = 1/16$, $B = 3/8$, $C = -1/160$, $D = -1/15$, $E = -8/15$, $F = 1/96$, $G = 11/96$. Now we express

$$\frac{-\frac{1}{15}(x+8)}{x^2+2x+2} = \frac{-\frac{1}{30}(2x+2)-\frac{7}{15}}{x^2+2x+2} = \frac{-\frac{1}{30}(2x+2)-\frac{7}{15}}{(x+1)^2+1}$$

and

$$\frac{\frac{1}{96}(x+11)}{x^2+2x+5} = \frac{\frac{1}{192}(2x+2)+\frac{10}{96}}{x^2+2x+5} = \frac{\frac{1}{192}(2x+2)+\frac{5}{48}}{(x+1)^2+4}.$$

Thus the integral becomes

$$\int\left(\frac{1/16}{x+1}+\frac{3/8}{(x+1)^2}-\frac{1/160}{x-1}-\frac{\frac{1}{30}(2x+2)+\frac{7}{15}}{(x+1)^2+1}+\frac{\frac{1}{192}(2x+2)+\frac{5}{48}}{(x+1)^2+4}\right)\mathrm{d}x$$

$$= \frac{1}{16}\ln|x+1| - \frac{3}{8}\frac{1}{x+1} - \frac{1}{160}\ln|x-1| - \frac{1}{30}\ln\left(x^2+2x+2\right) - \frac{7}{15}\arctan(x+1)$$

$$+\frac{1}{192}\ln\left(x^2+2x+5\right)+\frac{5}{96}\arctan\left(\frac{x+1}{2}\right)+C.$$

Finally, the appearance of repeated quadratic factors (which do not have real factors) poses a slightly more testing problem. Let us suppose that $Q(x)$ contains a factor $(x^2+2bx+c)^n$ (with $c-b^2>0$), for some integer $n>1$, then an expansion in partial fractions, followed by division, will produce a new type of term:

$$\frac{px+q}{(x^2+2bx+c)^n}.$$

It is immediately clear that, if we now write

$$px+q=\tfrac{1}{2}p(2x+2b)+q-pb,$$

then the only term that requires any discussion is

$$\int\frac{q-pb}{(x^2+2bx+c)^n}\,\mathrm{d}x.$$

This integral is most neatly tackled by constructing a reduction formula for it. We set

$$\begin{aligned}I_n&=\int\frac{\mathrm{d}x}{(x^2+2bx+c)^n}\\&=\frac{x+b}{(x^2+2bx+c)^n}-\int(x+b)\frac{-n(2x+2b)}{(x^2+2bx+c)^{n+1}}\,\mathrm{d}x,\end{aligned}$$

where we choose to use $(x+b)$ as the primitive of $\int 1.\mathrm{d}x$, and so we have

$$\begin{aligned}I_n&=\frac{x+b}{(x^2+2bx+c)^n}+2n\int\frac{x^2+2bx+c+b^2-c}{(x^2+2bx+c)^{n+1}}\,\mathrm{d}x\\&=\frac{x+b}{(x^2+2bx+c)^n}+2nI_n+2n(b^2-c)I_{n+1}\end{aligned}$$

$$\text{i.e. } 2n(b^2-c)I_{n+1}=(1-2n)I_n-\frac{x+b}{(x^2+2bx+c)^n}.$$

So, for example, we have

$$I_2=\int\frac{\mathrm{d}x}{(x^2+2bx+c)^2}=\frac{1}{2(c-b^2)}\left\{I_1+\frac{x+b}{x^2+2bx+c}\right\}$$

$$\text{where } I_1=\int\frac{\mathrm{d}x}{x^2+2bx+c}=\int\frac{\mathrm{d}x}{(x+b)^2+(c-b^2)}=\arctan\left(\frac{x+b}{\sqrt{c-b^2}}\right)$$

is a primitive. We use this result in the next example.

Example 19

Find $\displaystyle\int \frac{x^2 - x + 1}{(x-3)(x^2+4x+13)^2}\,\mathrm{d}x$.

The partial-fraction form of the integrand is

$$\frac{A}{x-3} + \frac{Bx+C}{x^2+4x+13} + \frac{Dx+E}{(x^2+4x+13)^2},$$

which requires $A = 7/1156$, $B = -7/1156$, $C = -49/1156$, $D = 27/34$, $E = 19/34$. Then we write the second and third terms as

$$-\frac{7}{1156}\frac{\frac{1}{2}(2x+4)+5}{x^2+4x+13} + \frac{1}{34}\frac{\frac{27}{2}(2x+4)-35}{(x^2+4x+13)^2}$$

with $\displaystyle\int \frac{\mathrm{d}x}{(x^2+4x+13)^2} = \frac{1}{18}\left\{\frac{x+2}{x^2+4x+13} + \arctan\left(\frac{x+2}{3}\right)\right\}$,

from the result obtained above. Thus the integral is

$$\tfrac{7}{1156}\ln|x-3| - \tfrac{7}{2312}\ln\left(x^2+4x+13\right) - \frac{1}{612}.\frac{35x+313}{x^2+4x+13} - \tfrac{455}{15606}\arctan\left(\frac{x+2}{3}\right) + C,$$

where we have suppressed some of the arithmetic.

Comment: The previous example is obviously somewhat involved, mainly because of the cumbersome coefficients (even though the coefficients in the original problem are quite simple – this is a typical complication in these problems). The main purpose of this example is to demonstrate that such integrals are susceptible to the methods that we have been describing.

Exercises 2

Find these definite and indefinite integrals:

$$\text{(a)} \int \frac{x^2-7x+3}{x-2}\,\mathrm{d}x\,; \text{ (b)} \int \frac{8x^4+3x^3-x^2+5x-2}{(2x+1)^2}\,\mathrm{d}x\,; \text{ (c)} \int_{-1}^{1} \frac{x^3-2x+2}{x^2+x-6}\,\mathrm{d}x\,;$$

$$\text{(d)} \int \frac{x^2+4x-5}{x^2-6x+13}\,\mathrm{d}x\,; \text{ (e)} \int \frac{3x-2}{x^3-x^2-15x-25}\,\mathrm{d}x\,; \text{ (f)} \int_{0}^{2} \frac{x^4-x^2+1}{2x^3-5x^2-24x+63}\,\mathrm{d}x\,.$$

3 The integration of trigonometric functions

In this chapter, we will discuss three important types of integral, typified by

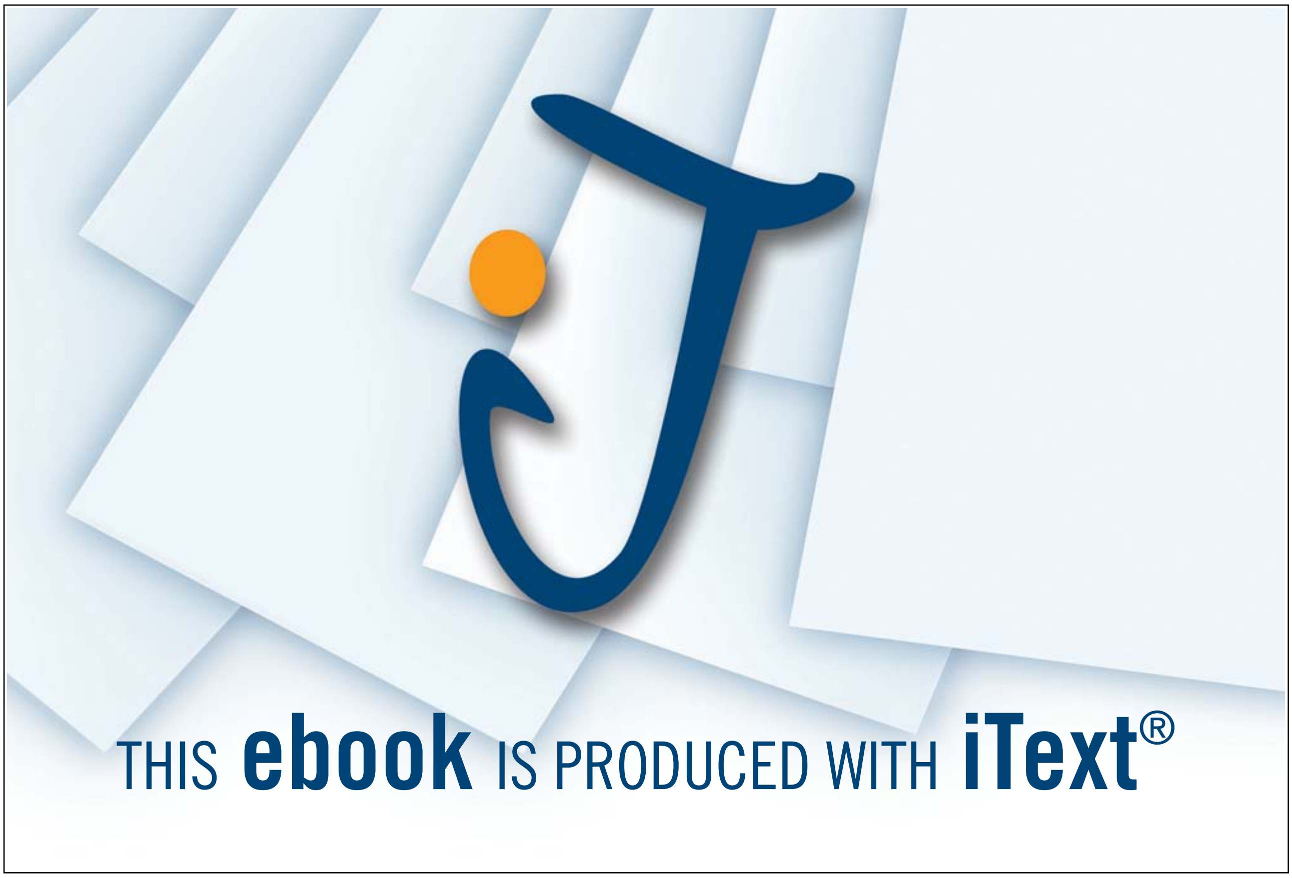
$$\int \sin 2x \cos 3x \,\mathrm{d}x \,;\ \int \sin^2 x \cos^3 x \,\mathrm{d}x \,;\ \int f(\sin x, \cos x)\,\mathrm{d}x \,,$$

where $f(\sin x, \cos x)$ is a rational function of its arguments. Of course, more complicated examples exist (some of which can be integrated using conventional methods) but those shown above are the ones most commonly encountered. Nevertheless, they are sufficient to allow us to describe the standard techniques for tackling all such problems.

3.1 Simple products

The products $\sin mx \cos nx$, $\sin mx \sin nx$ and $\cos mx \cos nx$ ($m \neq n$) are functions that can be integrated routinely, once an appropriate identity is invoked. Thus with

$$\sin mx \cos nx = \tfrac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right];$$



$$\sin mx \sin nx = \tfrac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right];$$

$$\cos mx \cos nx = \tfrac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right],$$

we immediately reduce these problems to elementary integrals.

Example 20

Find $\int \sin 2x \cos 3x \,\mathrm{d}x$.

This integral becomes

$$\tfrac{1}{2}\int\left(\sin 5x - \sin x\right)\mathrm{d}x = -\tfrac{1}{10}\cos 5x + \tfrac{1}{2}\cos x + C\,.$$

Comment: The cases where $m = n$ also follow directly, for then we use

$$\sin mx \cos mx = \tfrac{1}{2}\sin 2mx\,;\ \sin^2 mx = \tfrac{1}{2}\left(1-\cos 2mx\right);\ \cos^2 mx = \tfrac{1}{2}\left(1+\cos 2mx\right).$$

Example 21

Find the value of $\displaystyle\int_0^{\pi/2}\left(\sin 2x - \sin 4x\right)\sin 2x\,\mathrm{d}x$.

First we write

$$\left(\sin 2x - \sin 4x\right)\sin 2x = \tfrac{1}{2}\left(1-\cos 4x\right) - \tfrac{1}{2}\left(\cos 2x - \cos 6x\right),$$

and then integration yields the result

$$\left[\tfrac{1}{2}x - \tfrac{1}{8}\sin 4x - \tfrac{1}{4}\sin 2x + \tfrac{1}{12}\sin 6x\right]_0^{\pi/2} = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

3.2 Powers of sin and cos

We now turn to the more complicated integrals of the type

$$\int \sin^m x \cos^n x \,\mathrm{d}x\,,$$

which is best analysed *via* a reduction formula. Let

$$I_{m,n} = \int \sin^m x \cos^n x \, \mathrm{d}x$$

and write $\sin^m x \cos^n x = \left(\sin^m x \cos x\right)\cos^{n-1} x$ (but we could equally construct $\sin^{m-1} x\left(\cos^n x \sin x\right)$); thus we obtain

$$\begin{aligned}
I_{m,n} &= \int \left(\sin^m x \cos x\right)\cos^{n-1} x \, \mathrm{d}x \\
&= \tfrac{1}{m+1}\sin^{m+1} x \cos^{n-1} x - \int \frac{\sin^{m+1} x}{m+1}.(n-1)\cos^{n-2} x(-\sin x)\,\mathrm{d}x \\
&= \left(\tfrac{1}{m+1}\right)\sin^{m+1} x \cos^{n-1} x + \left(\tfrac{n-1}{m+1}\right)\int \sin^{m+2} x \cos^{n-2} x \, \mathrm{d}x \, .
\end{aligned}$$

This has generated an equation relating $I_{m,n}$ and $I_{m+2,n-2}$, but it is usually more convenient to write

$$\sin^{m+2} x \cos^{n-2} x = \sin^m x(1-\cos^2 x)\cos^{n-2} x = \sin^m x \cos^{n-2} x - \sin^m x \cos^n x$$

which leads to

$$I_{m,n} = \left(\tfrac{1}{m+1}\right)\sin^{m+1} x \cos^{n-1} x + \left(\tfrac{n-1}{m+1}\right)\left(I_{m,n-2} - I_{m,n}\right)$$

or $$I_{m,n} = \left(\tfrac{1}{m+n}\right)\left\{\sin^{m+1} x \cos^{n-1} x + (n-1)I_{m,n-2}\right\}.$$

This reduction formula is valid for any *m* and *n* (for which the integrals exist), but it is plainly a particularly useful identity if *m* and *n* are integers. In this case, and with *n* odd for *m* any integer, we have developed a powerful result, as the following example will demonstrate.

Example 22

Find $\int \sin^{10} x \cos^5 x \, \mathrm{d}x$..

We have $I_{10,5} = \tfrac{1}{15}\left(\sin^{11} x \cos^4 x + 4I_{10,3}\right)$

and then

$$I_{10,3} = \tfrac{1}{13}\left(\sin^{11} x \cos^2 x + 2I_{10,1}\right) \text{ where } I_{10,1} = \int \sin^{10} x \cos x \, \mathrm{d}x = \tfrac{1}{11}\sin^{11} x$$

(as a primitive). Thus we have

$$\int \sin^{10} x \cos^5 x \, dx = \tfrac{1}{15}\sin^{11} x \cos^4 x + \tfrac{4}{15.13}\sin^{11} x \cos^2 x + \tfrac{8}{15.13.11}\sin^{11} x + C$$

$$= \tfrac{1}{15}\sin^{11} x\left(\cos^4 x + \tfrac{4}{13}\cos^2 x + \tfrac{8}{13.11}\right) + C .$$

Comment: It should be clear that, if n is even, for any m, then the procedure is not quite so straightforward: the reduction formula will produce the integral

$$\int \sin^m x \, dx .$$

However, this is still a routine problem if m is odd, for then we simply set $\sin^m x = \sin^{m-1} x \sin x$ and then use $\sin^2 x = 1 - \cos^2 x$ as many times as is necessary; indeed, if n is not too large, this manoeuvre can be used immediately, without recourse to the reduction formula.

Example 23

Find the value of $\int_0^{\pi/2} \sin^5 x \cos^2 x \, dx$ ..

We write

$$\sin^5 x\cos^2 x = \left(\sin^2 x\right)^2 \sin x\cos^2 x$$
$$= \left(1-\cos^2 x\right)^2 \cos^2 x\sin x = \left(\cos^2 x - 2\cos^4 x + \cos^6 x\right)\sin x$$

which can now be integrated to give

$$\left[-\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x\right]_0^{\pi/2} = 0 - \left[-\tfrac{1}{3}+\tfrac{2}{5}-\tfrac{1}{7}\right] = \tfrac{8}{3.5.7}.$$

However, if m is even, then we must take more care; let us write

$$J_m = \int \sin^m x\,\mathrm{d}x$$

and then use

$$\sin^m x = \sin^{m-1} x\sin x.$$

Integrating by parts then gives

$$\begin{aligned} J_m &= -\cos x\sin^{m-1} x - \int(-\cos x)(m-1)\sin^{m-2} x\cos x\,\mathrm{d}x \\ &= -\cos x\sin^{m-1} x + (m-1)\int \sin^{m-2} x\cos^2 x\,\mathrm{d}x \\ &= -\cos x\sin^{m-1} x + (m-1)\int\left(\sin^{m-2} x - \sin^m x\right)\mathrm{d}x \\ &= -\cos x\sin^{m-1} x + (m-1)\left(J_{m-2} - J_m\right) \end{aligned}$$

i.e. $$J_m = \tfrac{1}{m}\left[-\cos x\sin^{m-1} x + (m-1)J_{m-2}\right].$$

(Again, we observe that m need not necessarily be an integer – even or odd – in this reduction formula.) If m is even, then we can reduce J_m to the determination of $J_0 = \int 1.\mathrm{d}x = x$.

Example 24

Find $\int \sin^4 x\,\mathrm{d}x$.

We have $J_4 = \int \sin^4 x\,\mathrm{d}x = \tfrac{1}{4}\left(-\cos x\sin^3 x + 3J_2\right)$

and $J_2 = \tfrac{1}{2}\left(-\cos x\sin x + J_0\right)$ with $J_0 = x$ (a primitive);

thus we obtain

$$\int \sin^4 x\,\mathrm{d}x = -\tfrac{1}{4}\cos x \sin^3 x - \tfrac{3}{8}\cos x \sin x + \tfrac{3}{8}x + C .$$

3.3 Rational functions of sin and cos

We complete this investigation of elementary integrals by examining the class of problems that arise from the integration of rational functions of *sin* and *cos*. All such integrals can be reduced to integrals of rational functions, and so all may be integrated by employing the methods described in Chapter 2. The transformation (substitution) that we adopt is an important one that holds a central position within the standard methods of integration: we use tangents of half angles.

Write $t = \tan(x/2)$, then we have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \tfrac{1}{2}\sec^2(x/2) = \tfrac{1}{2}\left(1+t^2\right)$$

with $$\sin x = 2\sin(x/2)\cos(x/2) = 2\frac{\tan(x/2)}{\sec^2(x/2)} = \frac{2t}{1+t^2}$$

and $$\cos x = 2\cos^2(x/2) - 1 = \frac{2-\sec^2(x/2)}{\sec^2(x/2)} = \frac{1-t^2}{1+t^2}.$$

Now since we start with the integral of a rational function of *sin* and *cos*, $\int f(\sin x, \cos x)\,\mathrm{d}x$, and the transformation that we propose introduces only rational terms (in *t*), we will necessarily produce a rational function of *t* to integrate. Note, however, that although this is a general method, which will demonstrate that all problems of this type can be reduced to one involving 'standard' integration methods, particular problems may be solved by a more direct route. We present two examples.

Example 25

Find $\displaystyle\int \frac{\mathrm{d}x}{5+4\cos x}$.

We introduce $t = \tan(x/2)$, as described above, which gives

$$\int \frac{\mathrm{d}x}{5+4\cos x} = \int \frac{2\mathrm{d}t/(1+t^2)}{5+4\left(\dfrac{1-t^2}{1+t^2}\right)} = 2\int \frac{\mathrm{d}t}{9+t^2};$$

to complete the calculation, we set $t = 3\tan u$, and so we obtain

$$\int \frac{\mathrm{d}x}{5+4\cos x} = 2\int \frac{3\sec^2 u}{9\sec^2 u}\,\mathrm{d}u = \tfrac{2}{3}\int \mathrm{d}u = \tfrac{2}{3}u + C$$
$$= \tfrac{2}{3}\arctan(t/3) + C = \tfrac{2}{3}\arctan\left(\tfrac{1}{3}\tan(x/2)\right) + C\,.$$

Example 26

Find $\displaystyle\int \frac{\sin^2 x}{(2+\cos x)^2}\,\mathrm{d}x$.

Introduce $t = \tan(x/2)$, then the integral becomes

$$\int \frac{\sin^2 x}{(2+\cos x)^2}\,\mathrm{d}x = \int \frac{\left[2t/(1+t^2)\right]^2 2\mathrm{d}t/(1+t^2)}{\left(2+\dfrac{1-t^2}{1+t^2}\right)^2}$$

$$= \int \frac{8t^2}{(1+t^2)(3+t^2)^2}\,\mathrm{d}t\,.$$

Now we follow the procedures of Chapter 2: we write

$$\frac{t^2+1-1}{(1+t^2)(3+t^2)^2} = \frac{1}{(3+t^2)^2} - \left\{\frac{A}{1+t^2} + \frac{B}{3+t^2} + \frac{C}{(3+t^2)^2}\right\}$$

which requires $A = -B = 1/4$ and $C = -1/2$, and so we obtain

$$\int \frac{\sin^2 x}{(2+\cos x)^2}\,\mathrm{d}x = 2\int \left\{\frac{6}{(3+t^2)^2} + \frac{1}{3+t^2} - \frac{1}{1+t^2}\right\}\mathrm{d}t\ .$$

In the terms containing $1/(3+t^2)$, we set $t = \sqrt{3}\tan u$, to produce

$$\begin{aligned}
2\int &\left(\frac{6}{9\sec^4 u} + \frac{1}{3\sec^2 u}\right)\sqrt{3}\sec^2 u\,\mathrm{d}u - 2\int \frac{\mathrm{d}t}{1+t^2} \\
&= \tfrac{4}{\sqrt{3}}\int \cos^2 u\,\mathrm{d}u + \tfrac{2}{\sqrt{3}}\int \mathrm{d}u - 2\int \frac{\mathrm{d}t}{1+t^2} \\
&= \tfrac{2}{\sqrt{3}}\int (1+\cos 2u)\,\mathrm{d}u + \tfrac{2}{\sqrt{3}}u - 2\arctan t \\
&= \tfrac{2}{\sqrt{3}}\left(u + \tfrac{1}{2}\sin 2u\right) + \tfrac{2}{\sqrt{3}}u - 2\arctan t + C \\
&= \tfrac{4}{\sqrt{3}}\arctan\left(t/\sqrt{3}\right) + \tfrac{1}{\sqrt{3}}\sin\left\{2\arctan\left(t/\sqrt{3}\right)\right\} - 2\arctan t + C \\
&= \tfrac{4}{\sqrt{3}}\arctan\left[\tfrac{1}{\sqrt{3}}\tan(x/2)\right] + \tfrac{1}{\sqrt{3}}\sin\left\{2\arctan\left[\tfrac{1}{\sqrt{3}}\tan(x/2)\right]\right\} - x + C\ .
\end{aligned}$$

Comment: The term in $\sin\{\ldots\}$ here can be written in a number of ways e.g.

$$\tfrac{1}{\sqrt{3}}\sin\left\{2\arctan\left[\tfrac{1}{\sqrt{3}}\tan(x/2)\right]\right\} = \frac{2\tan(x/2)}{3+\tan^2(x/2)}$$

which is probably to be preferred.

Exercises 3

1. Find these definite and indefinite integrals:

(a) $\int \sin 5x \cos 4x \, \mathrm{d}x$; (b) $\int_0^{\pi/2} (\cos 3x + \sin 2x) \cos 3x \, \mathrm{d}x$; (c) $\int \sin^5 x \cos^4 x \, \mathrm{d}x$;

(d) $\int \sin^4 x \cos^2 x \, \mathrm{d}x$; (e) $\int_0^{\pi} \frac{\mathrm{d}x}{3 + 2\cos x}$; (f) $\int \left(\frac{3 - \cos x}{2 + \sin x} \right) \mathrm{d}x$.

2. Find formulae for $\int_0^{\pi/2} \sin^n x \, \mathrm{d}x$ and $\int_0^{\pi/2} \cos^n x \, \mathrm{d}x$ where n is a positive integer, even or odd.

3. Show that $\int_0^1 \frac{x^4 (1-x)^4}{1+x^2} \mathrm{d}x = \frac{22}{7} - \pi$ (a great result!).

Answers

Exercises 1

1. (a) $-\frac{1}{50}(3-5x)^{10}+C$; (b) $-\frac{1}{2}\ln\left|\cos(2x-1)\right|+C$; (c) $-\ln\left(\sqrt{5}-2\right)-\ln\left(1+\sqrt{2}\right)$;

   (d) $-x^2\cos x+2\cos x+2x\sin x+C$; (e) $\dfrac{\pi}{2}-1$; (f) 1; (g) does not exist;

   (h) $\frac{1}{2}\exp\left(x^2\right)+C$; (i) $\ln 3$.

2. (a) $(1+\alpha)I_n = x^{\alpha+1}(\ln x)^n - nI_{n-1}$; (b) $I_n = \left(\frac{n-2}{n-1}\right)I_{n-2}+\left(\frac{1}{n-1}\right)\sec^{n-2}x\tan x$.

Exercises 2

(a) $\frac{1}{2}x^2-5x-7\ln\left|x-2\right|+C$; (b) $\frac{2}{3}x^3-\frac{5}{8}x^2+\frac{1}{2}x+\frac{17}{16}\ln\left|1+2x\right|+\dfrac{37}{16(1+2x)}+C$;

(c) $-2+\frac{19}{5}\ln 2-\frac{6}{5}\ln 3$; (d) $x+5\ln(x^2-6x+13)+6\arctan\left(\dfrac{x-3}{2}\right)+C$;;

(e) $\frac{13}{50}\ln\left|x-5\right|-\frac{13}{100}\ln(x^2+4x+5)+\frac{59}{50}\arctan(x+2)+C$;

(f) $\frac{565}{78}+\frac{2221}{1352}\ln(11/7)-\frac{1180}{169}\ln 3$.

Exercises 3

1. (a) $-\frac{1}{18}\cos 9x-\frac{1}{2}\cos x+C$; (b) $\dfrac{\pi}{4}-\dfrac{2}{5}$; (c) $-\left(\frac{1}{9}\sin^4 x+\frac{4}{63}\sin^2 x+\frac{8}{315}\right)\cos^5 x+C$;;

(d) $-\left(\frac{1}{9}\sin^4 x+\frac{4}{63}\sin^2 x+\frac{8}{315}\right)\cos^5 x+C$;; (e) $\dfrac{\pi}{\sqrt{5}}$;

(f) $2\ln(\sec(x/2))-\ln\left(1+\tan(x/2)+\tan^2(x/2)\right)+2\sqrt{3}\arctan\left[\frac{1}{\sqrt{3}}\left(1+2\tan(x/2)\right)\right]+C$.

2. Both can be expressed as $\dfrac{\sqrt{\pi}}{2}\dfrac{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}$ where Γ is the gamma function, where

$$\Gamma(N+1)=N!\ \text{and}\ \Gamma\left(N+\tfrac{1}{2}\right)=\frac{\sqrt{\pi}}{2^N}.1.3.5\ldots(2N-1)\text{, for } N=1,2,\ldots\ .$$

Part II

The integration of ordinary differential equations

List of Equations

This is a list of the types of equation, and specific examples, whose solutions are discussed. (Throughout, we write $y = y(x)$ and a prime denotes the derivative.)

Preface

This text is intended to provide an introduction to the methods for solving ordinary differential equations (ODEs). The material covered includes all the ideas that are discussed in any basic course at university, but excludes the methods of series solution. In addition, the presentation is intended to provide a general introduction to the relevant ideas, rather than as a text to be associated with any specific module or type of module. Indeed, the aim is to present the material in a way that enhances the understanding of the topic, and so can be used as an adjunct to a number of different modules – or simply to help the reader gain a broader experience. The plan is to go beyond the methods and techniques that are typically presented, but all the standard ideas are discussed here (and can be accessed through the comprehensive index).

It is assumed that the reader has a basic knowledge of, and practical experience in, the differential and the integral calculus; that is, minimally, something equivalent to a good pass in A-level Pure Mathematics, provided that the corresponding skills are firmly in place. An understanding of the techniques of integration, as developed in Part I, will prove very useful. We do not attempt to include any applications of the differential equations (e.g. in physics or engineering); this is properly left to a specific course of study that might be offered in a conventional applied mathematics or engineering mathematics or physics programme.

The approach adopted here is to present some general ideas, which might involve a notation, or a definition, or a theorem, or a classification, but most particularly methods of solution, explained with a number of carefully worked examples (there are 34 in total). A small number of exercises, with answers, are also offered, for the interested reader.

1 What is a differential equation ?

In this chapter we will briefly describe the nature of an ordinary differential equation (ODE), and what it means to solve it. Further, we will also mention the different types of solution that may arise. The classification of ODEs will be given, and with it an overview of the various types of equation that we will study in this notebook.

1.1 The nature and solution of differential equations

A differential equation is a relation, but more usually an algebraic equation, connecting a function, $y(x)$, and one or more derivatives of y. Thus relations (equations) such as

$f(x, y, y') = 0$, for example: $y' + xy^2 - x = 0$,

or $f(x, y, y', y'') = 0$, for example: $xy'' + (1-x)y' + 2y - x^2 = 0$,

where the prime denotes the derivative e.g. $\dfrac{\mathrm{d}y}{\mathrm{d}x} \equiv y'$, are *differential equations*. Any function, $y(x)$, which satisfies the equation is a solution, and any solution which contains as many independent, arbitrary constants as the highest derivative in the equation is the *general solution*. Thus the first example above will have a general solution which contains one arbitrary constant, and the second will have two. (Important and special, exceptional solutions – *singular* solutions – will be mentioned later.)

In the final analysis, any function – which may be defined via a relation – which has been proposed as a solution *exists* where it is *defined*, provided that it also satisfies the original differential equation. By 'defined' we mean that the function is to remain bounded and, in the context that we describe here, also to remain real. That the function is a solution is simply confirmed by direct substitution; if there is any doubt on this score, a suitable check should be performed. Indeed, because some equations may possess exceptional solutions, this – often routine – calculation becomes a necessary exercise in special cases.

1.2 Classification of ODEs

Ordinary differential equations are classified according to the highest derivative that appears in the equation, how this derivative appears and how the other terms in the equation appear. Often the correct identification of the classification (and then of the type, as we describe later) provides the first step in finding a solution of the equation. Let us first give the relevant definitions.

The *order* of the equation is the highest derivative in the equation (and so this will also specify the number of independent arbitrary constants that must appear in the general solution). The *degree* of the equation is the highest power (i.e. degree) of the highest derivative in the equation (after all the derivative terms have been rationalised e.g. $(y')^{3/2} = y^{1/3}$ must be regarded as $(y')^3 = y^{2/3}$ so of degree 3). A *linear* differential equation is one of the form

$$L(y) = g(x),$$

where L is a differential operator with the property that, for general y_1 and y_2,

$$L(y) = g(x),$$

An example of such an operator is

$$L \equiv \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}}{\mathrm{d}x} + (3+x)$$

which gives the equation $y'' + 2xy' + (3+x)y = g(x)$.

We see that the operator L on y does not itself contain y, although it may contain x explicitly (in the coefficients).

Here are a few examples that should help to explain this classification.

(a) $y' + xy = x^2$: order 1, degree 1, linear;

(b) $y' + xy^2 = x^2$: order 1, degree 1, nonlinear;

(c) $(y')^2 + xy = \sin x$: order 1, degree 2;

(d) $xy'' + (1+x)y' + 2y = x^2$: order 2, degree 1, linear;

(e) $y'' + y' + \sin y = x$: order 2, degree 1, nonlinear;

(f) $(y'')^3 + (y')^2 + y = 3$: order 2, degree 3.

Finally, we may also, for a linear ODE – and this applies *only* to linear equations – describe the functions that contribute to the general solution; this important idea is one which, as we shall see later, helps us to seek solutions. Consider the linear ODE

$$L(y) = g(x),$$

where $g(x)$ is given (and often called the *forcing* term); let us write the solution as the sum of two functions:

$$y(x) = y_I(x) + y_{II}(x),$$

so that $L(y) = L(y_I + y_{II}) = L(y_I) + L(y_{II}) = g(x)$.

We choose

$$L(y_I) = 0 \text{ and } L(y_{II}) = g(x),$$

where y_{II} is *any* function which satisfies the original equation; y_I is the most general solution of the *homogeneous* equation i.e. the one without a forcing term. (The formal definition of homogeneity will be given later; see §2.2.) It is usual to call y_I the *complementary function* or simply CF (and then we often write this solution as y_{CF}); the solution y_{II} is a *particular integral* or PI (and then, correspondingly, we write y_{PI}). The complete general solution is therefore

$$y = y_{CF} + y_{PI}.$$

1.3 Overview of the equations to be discussed

In Chapter 2 we will describe the methods of solution for the standard, first order ODEs that are commonly met in the early stages of modern mathematics programmes. In particular, we examine these three types of equation:

separable e.g. $y' = (1+x)(1+y)$;

homogeneous e.g. $y' = \dfrac{x^2 - y^2}{x^2 + y^2}$;

general linear of the form $y' + p(x)y = q(x)$.

This is followed, in Chapter 3, by a discussion of a number of other important

equations and special cases of some note. So we will find the solutions of these equations:

Bernoulli equation: $y' + p(x)y = q(x)y^n$;

Clairaut equation: $y = xy' + \alpha(y')$ where α is a given function of y', and its generalisation – the Lagrange equation: $y = x\beta(y') + \alpha(y')$;;

Riccati equation: $y' = a(x)y^2 + b(x)y + c(x)$.

We also consider the equations that are, or may be written as, exact differentials; finally the special cases of $f(x, y, y') = 0$, with either *x* or *y* absent, will be analysed.

In Chapter 4 we consider second-order, linear ODEs, and in particular those with constant coefficients, such as

$$y'' + 2y' + 3y = 0$$

and those of Euler type e.g.

$$x^2 y'' + 2xy' + 3y = 0.$$

We will also include the method of variation of parameters, and its simpler variant that allows for the reduction of the order of the equation. This latter method is particularly useful when we need to find a second solution of the equation or a particular integral (PI); methods for finding some PIs directly will also be developed.

Finally, in Chapter 5, we look at three ideas that play a rôle in any deeper study of ODEs. First, the manoeuvre of interchanging *x* and *y*, so that we treat the solution as $x(y)$ rather than $y(x)$; secondly, the occurrence of singular solutions and how to find them; thirdly, we conclude with a few comments on uniqueness, and invoke this property to generate two familiar trigonometric identities.

Exercises 1

Use the information in §1.2 to classify the following equations.

(a) $y' + x\sin y = \mathrm{e}^x$; (b) $xy' + x^2 y = 2$; (c) $(y')^2 + y^2 = 1$;;

(d) $y'' + y' + y = 0$; (e) $x^2 y'' + xy' + y = x^2$; (f) $xy'' + 2y' + y^2 = x\mathrm{e}^x$;

(g) $y''' + y'' + yy' + xy = \sin x$; (h) $y^{iv} + y'' + y = x$.

2 First order ODEs: standard results

We present the three standard and most common types of equation, and associated methods of solution, that are taught in introductory courses on ODEs. In each case we describe the underlying technique that is involved and present three worked examples; a few exercises are offered at the end of the chapter.

2.1 Separable equations

The simplest of all types of first order ODEs – because the problem immediately reduces to direct integration – takes the form

$$y' = f(x)g(y);$$

the expression on the right is a *product* of a function of *x* and a function of *y*. (This is what is meant by 'separable'.) Note that this form is equivalent, for suitable *f* and *g*, to

$$y' = \frac{f(x)}{g(y)} \quad \left(= f(x).\frac{1}{g(y)} \right) \text{ and } y' = \frac{g(y)}{f(x)} \quad \left(= \frac{1}{f(x)}.g(y) \right).$$

To proceed, we simply write

$$\frac{y'}{g(y)} = f(x) \text{ (for } g(y) \neq 0 \text{)}$$

and then integrate each side of this equation with respect to *x*:

$$\int \frac{y'}{g(y)} \mathrm{d}x = \int f(x)\,\mathrm{d}x \text{ and so } \int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x.$$

The process of integration will generate one arbitrary constant – not two, because each integration produces an additive arbitrary constant which can be combined into a single one. Any exceptional solutions, if they exist, that are associated with division by zero (here $g(y) = 0$) can be investigated separately (and, more often than not, they pose no difficulties).

Example 1

Find the general solution of the equation $y' = 2x\left(1 + y^2\right)$

We write $\dfrac{y'}{1 + y^2} = 2x$ (and we note that $1 + y^2 \geq 1$: there can be no division by zero), and so

$$\int \frac{\mathrm{d}y}{1 + y^2} = \int 2x\,\mathrm{d}x \text{ i.e. } \arctan(y) = x^2 + A,$$

where A is an arbitrary constant. The general solution is therefore

$$y(x) = \tan\left(A + x^2\right).$$

Example 2

Find the general solution of the equation $\left(1+\mathrm{e}^{-x}\right)yy' = 1,$ and then that solution which satisfies $y(0) = 1.$

We write $yy' = \dfrac{1}{1+\mathrm{e}^{-x}} = \dfrac{\mathrm{e}^x}{1+\mathrm{e}^x}$ (and $1+\mathrm{e}^{-x} > 1$ for $\forall\, x$) and so we obtain

$$\tfrac{1}{2}y^2 = \ln\left|1+\mathrm{e}^x\right| + \tfrac{1}{2}A = \ln\left(1+\mathrm{e}^x\right) + \tfrac{1}{2}A,$$

where A is an arbitrary constant. The general solution is

$$y(x) = \pm\sqrt{A + 2\ln\left(1+\mathrm{e}^x\right)}.$$

Now $y(0) = 1$ requires that

$$1 = \pm\sqrt{A + 2\ln 2}$$

which is satisfied only if we select $A = 1 - 2\ln 2$ and then use the positive sign; the resulting solution is therefore

$$y(x) = \sqrt{1 + 2\ln\left[\tfrac{1}{2}\left(1 + \mathrm{e}^x\right)\right]}.$$

Example 3

Find the general solution of the equation $\left(1 - \mathrm{e}^x\right)y' \sec^2 y + 2\mathrm{e}^x \tan y = 0.$

Here we write the equation as $\dfrac{\sec^2 y}{\tan y} y' = -\dfrac{2\mathrm{e}^x}{1 - \mathrm{e}^x}$ (for $\tan y \neq 0$, $1 - \mathrm{e}^x \neq 0$), and so we obtain

$$\ln\left|\tan y\right| = 2\ln\left|1 - \mathrm{e}^x\right| + A \text{ or } \frac{\tan y}{\left(1 - \mathrm{e}^x\right)^2} = \pm\mathrm{e}^A = B,$$

where *A*, and then *B*, are arbitrary constants. Thus the general solution is

$$y(x) = \arctan\left[B\left(1 - \mathrm{e}^x\right)^2\right].$$

However, before we leave this example, we must examine the possibility that the conditions excluded when we divided might give rise to other solutions. If $\tan y = 0$, all the corresponding solutions ($y = n\pi, n = 0, \pm 1, \pm 2, \ldots$) arise from the choice $B = 0$, so these solutions are already included. The case $x = 0$ corresponds to $\tan y = 0$, and this is consistent with both the original equation and the solution. Thus both special solutions are valid and are recovered from the general solution.

Comment: Before we leave this type of ODE, we observe that an equation of the form

$$y' = f(ax + by + c),$$

where *a*, *b* and *c* are constants, can be recast as a separable equation by writing

$$u = ax + by + c \text{ to give } b^{-1}\left(u' - a\right) = f(u) \text{ or } u' = bf(u) + a,$$

which is clearly separable.

2.2 Homogeneous equations

We describe, first, the notion of homogeneity. A function, $f(x, y)$, is said to be *homogeneous of degree n* if

$$f(kx, ky) = k^n f(x, y) \quad (\forall k \neq 0).$$

A differential equation

$$y' = f(x,y)$$

is of *homogeneous type* if $f(x,y)$ is homogeneous of degree 0 e.g.

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \text{ gives } f(kx,ky) = \frac{k^2\left(x^2 - y^2\right)}{k^2\left(x^2 + y^2\right)} = f(x,y).$$

On the other hand, $f(x,y) = \dfrac{x^2}{x+y}$ is not:

$$f(kx,ky) = \frac{k^2x^2}{k(x+y)} = k\frac{x^2}{x+y} \text{ so it is of degree 1.}$$

[This same basic notion of homogeneity can be used to describe *linear* ODEs. Consider the linear, second order ODE

$$a(x)y'' + b(x)y' + c(x)y = g(x),$$

where the dependence on *x* is altogether immaterial. The equation with $g(x) \equiv 0$ is homogeneous of degree 1 in *y*; to see this, write *ky* for *y* to give

$$a(x)ky'' + b(x)ky' + c(x)ky = 0$$

i.e. $$k\left[a(x)y'' + b(x)y' + c(x)y\right] = 0$$

(and then we recover $a(x)y'' + b(x)y' + c(x)y = 0,$ since $k \neq 0$).

Thus the equation with no forcing term (i.e. zero right-hand side) is usually referred to as the *homogeneous equation*; note that this argument clearly fails if there is a right-hand side, independent of *y*, because then *k* cannot be factored out.]

We now return to the first-order ODE of homogeneous type:

$$y' = f(x,y) \text{ with } f(kx,ky) = f(x,y),$$

which clearly requires that the form of *f* is

$$\frac{\text{homogeneous function of degree } n}{\text{homogeneous function of degree } n},$$

and very often these two functions are simple polynomials (in *x* and *y*). The method of solution is based on the recognition that

$$f(kx,ky) = f(x,y) \;\Rightarrow\; f(x,y) = F(y/x),$$

for some F; thus we seek a solution for y/x rather than for y directly. We set $v(x) = y(x)/x$, so that $y(x) = xv(x)$, and we now solve for $v(x)$; once we have v, it is elementary to find y. Hence we obtain the equation

$$y' = v + xv' = F(v) \text{ i.e. } xv' = F(v) - v,$$

which is of separable form; see the previous section.

Example 4

Find the general solution of the equation $yy' = 2y - x$.

When written as $y' = \dfrac{2y-x}{y}$ we see that this equation is of homogeneous type, so we set (in the original equation) $y = xv$ to give

$$xv(v + xv') = x(2v-1) \text{ and so } xvv' = 2v - 1 - v^2 = -(v-1)^2 \; (x \neq 0).$$

We note, for v to be defined, that necessarily $x \neq 0$ and so the condition imposed to manipulate the equation does not constitute a further restriction. The resulting solution of the equation may, however, still be defined on $x = 0$; this is readily checked in particular cases. Thus we have

$$\int \frac{v\,\mathrm{d}v}{(v-1)^2} = -\int \frac{\mathrm{d}x}{x} \qquad (v \neq 1)$$

$$\text{or} \quad \int\left(\frac{1}{v-1}+\frac{1}{(v-1)^2}\right)\mathrm{d}v = -\int\frac{\mathrm{d}x}{x} \ \text{ and so } \ \ln|v-1| - \frac{1}{v-1} = -\ln|x| + A.$$

Thus we may write

$$\ln|x(v-1)| = A + \frac{1}{v-1} \ \text{ and so } \ y - x = B\exp\left(\frac{x}{y-x}\right),$$

where B is an arbitrary constant. Notice that the best we can obtain here is an *integral* of the original equation – not an explicit solution in the form $y = y(x)$; this is not untypical of solutions of homogeneous-type ODEs. Finally, if $v = 1$ i.e. $y = x,$ then the manipulation above is not valid, but this *is* a solution of the original equation and, furthermore, it is recovered from the general solution with the choice $B = 0.$

Example 5

Find all the solutions of the equation $xy' = y + \sqrt{y^2 - x^2}$.

This equation is of homogeneous type, so we write $y = xv$:

$$x(v + xv') = x\left(v + \sqrt{v^2 - 1}\right) \ \text{ or } \ xv' = \sqrt{v^2 - 1}$$

Thus we obtain

$$\ln\left|v + \sqrt{v^2 - 1}\right| = \ln|x| + A$$

$$\text{or} \quad \frac{v + \sqrt{v^2-1}}{x} = B \ \text{ i.e. } \ y + \sqrt{y^2 - x^2} = Bx^2$$

is the general solution, where B is an arbitrary constant. We see that the case $v = \pm 1$ i.e. $y = \pm x$ are two further solutions of the original differential equation, *but neither* can be recovered from the general solution for *any* choice of B. The case $x = 0$ simply gives $y = 0$ (a point). Hence the complete description of the solutions is

$$y(x) = \frac{1 + B^2x^2}{2B}; \ y(x) = \pm x.$$

Comment: In this example, we see that there is not a unique solution for all possible boundary conditions. So, if we are given $y(a) = a$, for any $a \neq 0,$ then both $y = x$ and $y = (a/2)\left[1 + (x/a)^2\right]$ (i.e. $B = 1/a$) are possible solutions.

Example 6

Find the general solution of the equation $x\left(1-\mathrm{e}^{-y/x}\right)y' = 2y + (x-y)\mathrm{e}^{-y/x}$

The presence of the exponential terms does not affect the homogeneity, so we set $y = xv$ (and note that the equation itself implies $x \neq 0$) to give

$$x\left(1-\mathrm{e}^{-v}\right)\left(v + xv'\right) = 2xv + x(1-v)\mathrm{e}^{-v}$$

or $$x\left(1-\mathrm{e}^{-v}\right)v' = v + \mathrm{e}^{-v}.$$

Thus we obtain

$$\int \frac{1-\mathrm{e}^{-v}}{v+\mathrm{e}^{-v}}\,\mathrm{d}v = \int \frac{\mathrm{d}x}{x}$$

(and $v + \mathrm{e}^{-v}$ is never zero; indeed, $v + \mathrm{e}^{-v} \geq 1$) and so

$$\ln\left(v + \mathrm{e}^{-v}\right) = \ln|x| + A \quad \text{or} \quad \frac{v + \mathrm{e}^{-v}}{x} = B$$

which produces the general integral of the equation in the form

$$y + x\mathrm{e}^{-y/x} = Bx^2.$$

2.3 The general linear equation

We now turn to the equation that is linear in y (and of the first order, of course); this equation takes the form

$$y' + p(x)y = q(x)$$

for general functions p and q. In order to solve this equation, we first consider the case with zero right-hand side i.e.

$$y' + p(x)y = 0;$$

this equation is separable, so we are able to solve directly:

$$\int \frac{\mathrm{d}y}{y} = -\int p(x)\,\mathrm{d}x$$

(for $y \neq 0$, although we observe that $y \equiv 0$ *is* a solution of our reduced equation). Thus

$$\ln|y| = -\int^x p(x')\,\mathrm{d}x' \quad \text{or} \quad y = A\exp\left\{-\int^x p(x')\,\mathrm{d}x'\right\}$$

and the details of the representation of this solution are irrelevant. We could, for example, dispense with the arbitrary constant A, because the integral is indefinite; on the other hand, we could write

$$y = A\exp\left\{-\int_0^x p(x')\,\mathrm{d}x'\right\}$$

– or use any other suitable fixed lower limit – and then the use of A is necessary. The important property to note is that

$$y\exp\left\{\int^x p(x')\,\mathrm{d}x'\right\} = \text{constant}$$

and so, after differentiating this expression once, we obtain

$$y'\exp\left\{\int^x p(x')\,\mathrm{d}x'\right\} + yp(x)\exp\left\{\int^x p(x')\,\mathrm{d}x'\right\} \quad (=0)$$
$$= \left[y' + p(x)y\right]\exp\left\{\int^x p(x')\,\mathrm{d}x'\right\}$$

which is the left-hand side of the equation, multiplied by the exponential function. Thus, if we choose to follow this construction applied to our original equation (with the $q(x)$ retained and the exp factor can never be zero), we will obtain

$$[y' + p(x)y]\exp\left\{\int^x p(x')\,dx'\right\} = q(x)\exp\left\{\int^x p(x')\,dx'\right\}.$$

Note that we may accomplish this without actually performing the integration of $p(x)$ (although, in practice, this is usually possible and is therefore done). When the integration is possible, the technique requires only that a suitable function can be found which is proportional to $\exp\left\{\int^x p(x')\,dx'\right\}$, so any arbitrary constant – which would be multiplicative – is unnecessary.

From what has gone before, we see that we may now write

$$\frac{d}{dx}\left\{y\exp\left[\int^x p(x')\,dx'\right]\right\} = q(x)\exp\left\{\int^x p(x')\,dx'\right\}:$$

the left-hand side can be written as the derivative of a simple term – the product of $y(x)$ and a known function. The left-hand side has been converted into an *exact differential*. The function that makes this possible – the exponential here – is called an *integrating factor* because, of course, it leads directly to the equation being integrated completely. Performing this integration, and rearranging, gives

$$y(x) = \exp\left\{-\int^x p(x')\,dx'\right\}\int^x q(x')\exp\left\{\int^{x'} p(\hat{x})\,d\hat{x}\right\}dx',$$

although it is far from good practice to remember, quote and substitute into this formula. This construct does, however, demonstrate that all linear, first-order equations can be solved. (This general form of solution is accepted as *the solution*, even if the integrals cannot be expressed in terms of known functions; written like this, the solution is said to have been reduced to a *quadrature*, implying that a numerical method would need to be employed, in general, to take it any further.)

Example 7

Find the general solution of the equation $y' + \dfrac{3}{x}y = x$.

The integrating factor is obtained from

$$\exp\left(\int^x \frac{3}{x'}\,dx'\right) = \exp\left(3\ln|x| + \text{constant}\right) = A\exp\left(\ln\left|x^3\right|\right)$$

and it is sufficient to take simply x^3 as the integrating factor; thus we form

$$x^3\left(y' + \frac{3}{x}y\right) = x^4.$$

This becomes $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 y\right) = x^4$ and so $x^3 y = \frac{1}{5}x^5 + A$

or $y(x) = \frac{1}{5}x^2 + \frac{A}{x^3}$,

which is the general solution, with A the arbitrary constant.

Example 8

Find the general solution of the equation $\left(1+x^2\right)y' + 4xy = 2x$ and then that solution which satisfies $y(0) = 1$.

In order to identify the integrating factor, we must first write the equation as

$$y' + \frac{4x}{1+x^2}y = \frac{2x}{1+x^2}$$

and then we construct

$$\exp\left\{\int^x \frac{4x'}{1+x'^2}\,\mathrm{d}x'\right\} = \exp\left\{2\ln\left(1+x^2\right) + \text{constant}\right\} = A\left(1+x^2\right)^2.$$

Thus we choose to multiply the second version of the equation by $\left(1+x^2\right)^2$ (or the first version by $\left(1+x^2\right)$) to give

$$\left(1+x^2\right)^2 y' + 4x\left(1+x^2\right)y = 2x\left(1+x^2\right)$$

so that $\frac{\mathrm{d}}{\mathrm{d}x}\left[y\left(1+x^2\right)^2\right] = 2x\left(1+x^2\right)$

or $y\left(1+x^2\right)^2 = x^2 + \frac{1}{2}x^4 + A$.

Thus the general solution is

$$y(x) = \frac{1}{2}\frac{x^2\left(2+x^2\right)}{\left(1+x^2\right)^2} + \frac{A}{\left(1+x^2\right)^2}$$

where A is the arbitrary constant.

Finally, the solution which satisfies $y(0) = 1$ must have $A = 1$, so the required solution is

$$y(x) = \frac{1}{2}\frac{x^2\left(2+x^2\right)}{\left(1+x^2\right)^2} + \frac{A}{\left(1+x^2\right)^2}$$

Example 9

Find the general solution of the equation $y' \sin x - y \cos x = 2 \sin 2x$.

First, we write

$$y' - y \cot x = \frac{2 \sin 2x}{\sin x} = 4 \cos x \quad (\sin x \neq 0)$$

and the integrating factor is

$$\exp\left\{-\int^x \cot x' \, \mathrm{d}x'\right\} = \exp\left\{-\ln\left|\sin x\right| + \text{constant}\right\}$$

and so we elect to multiply the second version of the equation by $1/\sin x$; this gives

$$\frac{1}{\sin x}y' - \frac{\cos x}{\sin^2 x}y = 4\frac{\cos x}{\sin x}$$

$$\text{i.e.}\quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sin x}\right) = 4\frac{\cos x}{\sin x}.$$

Thus we obtain $y/\sin x = 4\ln|\sin x| + A$

and so the general solution, with arbitrary constant *A*, is

$$y(x) = A\sin x + 4\sin x \ln|\sin x|$$

(which *is* defined as $|\sin x| \to 0$).

This concludes our discussion of the standard methods available for the solution of first order ODEs.

Exercises 2

Find all the solutions of these equations, writing these explicitly in the form $y = y(x)$ wherever possible.

$$\text{(a)}\ xy' + y\ln y = 0\,;\ \text{(b)}\ \left(1+x^2\right)\mathrm{e}^y y' = 2x\left(1+\mathrm{e}^y\right);\ \text{(c)}\ x(x-y)y' + y^2 = 0\,;$$

$$\text{(d)}\ \left[2x^2 + y^2\left(1+\mathrm{e}^{-(x/y)^2}\right)\right]y' - 2xy = 0\,;\ \text{(e)}\ xy' - 2y = x^3\cos x\,;$$

$$\text{(f)}\ \left(1+x^2\right)y' + (1-x)^2 y = x\mathrm{e}^{-x},$$

and for (d) also find that solution which satisfies $y(0) = 4$.

3 First order ODEs: special equations

We now turn to a consideration of some types of equation that do not fall into the categories discussed in Chapter 2 but which, nevertheless, can be solved.

3.1 The Bernoulli equation

This equation is the simplest that we consider in this chapter: it can be transformed into the standard linear form (§2.3). The most general form of the equation is

$$y' + p(x)y = q(x)y^n$$

where n is any constant (although we exclude $n = 0$ and $n = 1$ because, in these two cases, the equation is immediately a version of the linear equation). To see how we should proceed, we divide throughout by y^n:

$$y^{-n}y' + p(x)y^{1-n} = q(x) \quad (y \neq 0)$$

and we observe that the derivative of y^{1-n}, with respect to x, is $(1-n)y^{-n}y'$ which is essentially the first term in the equation. This suggests that we introduce $u(x) = y^{1-n}$, so that $u' = (1-n)y^{-n}y'$, and hence the equation becomes

$$\frac{1}{1-n}u' + p(x)u = q(x)$$

which is linear in $u(x)$, and so we may use the technique described in §2.3.

Example 10

Find the general solution of the equation $y' - xy = xy^3$.

We introduce $u = y^{-2}$ ($y \neq 0$, but we note that $y \equiv 0$ is obviously a solution of the given equation); thus $u' = -2y^{-3}y'$ and so we obtain

$$-\tfrac{1}{2}y^3u' - xy^3u = xy^3 \text{ or } \tfrac{1}{2}u' + xu = -x.$$

Writing this as $u' + 2xu = -2x$, we see that this equation has the integrating factor

$$\exp\left\{\int^x 2x'\,\mathrm{d}x'\right\} = A\exp\left(x^2\right)$$

and so we construct

$$\mathrm{e}^{x^2}\left(u' + 2xu\right) = -2x\mathrm{e}^{x^2} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(u\mathrm{e}^{x^2}\right) = -2x\mathrm{e}^{x^2}$$

and so we have $u\mathrm{e}^{x^2} = -\mathrm{e}^{x^2} + A.$

Thus the general solution for u is

$$u(x) = A\mathrm{e}^{-x^2} - 1,$$

where A is an arbitrary constant; then the required solution for y is

$$y(x) = \pm\frac{1}{\sqrt{A\mathrm{e}^{-x^2} - 1}},$$

and $y = 0$ is recovered when we allow $A \to +\infty$.

3.2 The Clairaut equation

The structure of this equation is, at first sight, considerably more involved than anything that we have met so far; it takes the form

$$y = xy' + \alpha(y')$$

where α is any function of y'. The method we adopt, in an effort to simplify the problem, is to differentiate the equation once (so we will assume that the solution, if one exists, has a continuous second derivative in some domain). This derivative of the equation then produces

$$y' = xy'' + y' + y''\alpha'(y') \quad \text{and so} \quad (x + \alpha')y'' = 0,$$

and this equation has solutions

$$y'' = 0 \text{ and } \frac{\mathrm{d}\alpha}{\mathrm{d}y'} = -x.$$

The former possibility shows that $y(x) = Ax + B,$ where A and B are arbitrary constants; however, in order to satisfy the original equation we must have

$$Ax + B = Ax + \alpha(A) \text{ i.e. } B = \alpha(A)$$

and so we obtain the general solution

$$y(x) = Ax + \alpha(A)$$

where A is the single – expected – arbitrary constant.

The second solution of the differentiated equation can be expressed as

$$x = -\alpha'(p), \quad y = -p\alpha'(p) + \alpha(p)$$

where the original version of the differential equation is written with $x = -\alpha'(p)$; p is a parameter here. We now have a second solution which has been written in parametric form; it contains no arbitrary constant and so is an exceptional solution.

Example 11

Find all the solutions of the equation $y = xy' + \dfrac{1}{4y'}$.

First we differentiate with respect to x to give

$$\left(x - \frac{1}{4(y')^2}\right)y'' = 0$$

which has one solution

$$y = Ax + B \text{ with } B = \frac{1}{4A} \text{ i.e. } y(x) = Ax + \frac{1}{4A}$$

where A is an arbitrary constant; this is the general solution.

A second solution is obtained from

$$x = \frac{1}{4p^2} \text{ with } y = xp + \frac{1}{4p} = \frac{1}{2p}$$

and here it is an elementary exercise to eliminate the parameter p, to give

$$y^2 = x$$

(which certainly cannot be obtained by any choice of the arbitrary constant A in the general solution). Thus we have the complete prescription of the solution as

$$y(x) = Ax + \frac{1}{4A}; \; y(x) = \pm\sqrt{x}\,.$$

Finally we comment that there is a generalisation of this equation – the *Lagrange equation* – which is

$$y = x\beta(y') + \alpha(y');$$

the same manoeuvre that we adopted for the Clairaut equation now gives

$$y' = \beta(y') + \left[x\beta'(y') + \alpha'(y')\right]y''.$$

One solution is $y'' = 0$, provided that there exists at least one real root of the equation $y' = \beta(y')$; if there are many roots, then there will be a corresponding set of straight lines

$$y = x\beta(p) + \alpha(p) = xp + \alpha(p)$$

where p is a (real) root of $p = \beta(p)$. Note that this solution (or set of solutions) does not contain an arbitrary constant, and so we have not generated the general solution.

A second solution of our equation follows, even though $y' \neq \beta(y')$; let us write

$$\left[x\beta'(y') + \alpha'(y')\right]\frac{\mathrm{d}y'}{\mathrm{d}x} = y' - \beta(y')$$

which can be expressed in the form

$$x\beta'(p)+\alpha'(p)=\left[p-\beta(p)\right]\frac{\mathrm{d}x}{\mathrm{d}p}$$

where we have written $p=y'$. This is a *linear* equation in $x(p)=x(y')$: we have interchanged the rôles of the dependent and independent variables ! (We shall write more of this idea later; see §5.1.) The upshot is that we may solve this equation (see §2.3) to give the solution in the form

$$x=x(p) \text{ with } y=x(p)\beta(p)+\alpha(p),$$

which provides a parametric representation of the solution – and because it involves one integration, it will contain one arbitrary constant: this is therefore the general solution.

Example 12

Find all the solutions of the equation $y=x+(y')^2\left(1-\tfrac{2}{3}y'\right)$.

This equation is of Lagrange type because of the term in x (which does not appear as xy'). First we differentiate with respect to x to give

$$y'=1+y'(2-2y')y'' \text{ or } (1-y')(1+2y'y'')=0,$$

and so one solution is $y''=0$ with $y'=1$ i.e. $y=x+C$; the original equation then requires that

$$x + C = x + \tfrac{1}{3} \text{ so } C = \tfrac{1}{3}: \; y(x) = x + \tfrac{1}{3}.$$

The other solution arises from

$$1 + 2y'y'' = 0 \text{ so } x + (y')^2 = A$$

where A is an arbitrary constant. This, together with the original equation, gives

$$x = A - p^2; \; y = A - p^2 + p^2\left(1 - \tfrac{2}{3}p\right) = A - \tfrac{2}{3}p^3$$

and in this case the parameter p can be eliminated:

$$\left[\tfrac{3}{2}(A - y)\right]^2 = (A - x)^3 \text{ or } y(x) = A \pm \tfrac{2}{3}\sqrt{(A - x)^3}$$

which is the general solution.

3.3 The Riccati equation

This is one of the more important first-order equations; it is of degree 1, but quadratic in y:

$$y' = a(x)y^2 + b(x)y + c(x).$$

The method of solution enables this nonlinear equation to be transformed into a *linear*, second-order equation. To accomplish this, let us seek a solution in the form

$$y(x) = f(x)\frac{u'}{u}$$

where $f(x)$ is to be chosen so that $u(x)$ will satisfy a second-order equation. We find y' and substitute to give

$$f'\frac{u'}{u} + f\left(\frac{u''}{u} - \frac{(u')^2}{u^2}\right) = af^2\left(\frac{u'}{u}\right)^2 + bf\frac{u'}{u} + c.$$

We now select $f(x)$ so that the terms $(u'/u)^2$ are eliminated: $f(x) = -1/a(x)$, which gives

$$\frac{a'}{a^2}\frac{u'}{u} - \frac{1}{a}\frac{u''}{u} = -\frac{b}{a}\frac{u'}{u} + c,$$

and then multiplication throughout by au produces

$$u'' - \left(b + \frac{a'}{a}\right)u' + acu = 0.$$

This is a linear, second-order ODE (with variable coefficients). The methods for solving this type of equation are well-known, even if it is necessary to seek a solution as an appropriate power series in some cases (e.g. via the method of Frobenius). Of course, in simple cases it may be possible to integrate in terms of elementary functions. The solution of the Riccati equation is then recovered from $y = -u'/(au)$.

Example 13

Find the general solution of the equation $x^2 y' = 8xy - x^2 y^2 - 20$.

First, we introduce $y(x) = -\left(\frac{-x^2}{x^2}\right)\frac{u'}{u} = \frac{u'}{u}$ and then the equation becomes

$$x^2\left(\frac{u''}{u} - \frac{(u')^2}{u^2}\right) = 8x\frac{u'}{u} - x^2\left(\frac{u'}{u}\right)^2 - 20.$$

This simplifies to produce

$$x^2 u'' - 8xu' + 20u = 0$$

and this type of equation has a particularly simple general solution (see §4.2). We set $u(x) = x^{\lambda}$ to give (for $x \neq 0$)

$$\lambda(\lambda - 1) - 8\lambda + 20 = 0 \text{ or } \lambda^2 - 9\lambda + 20 = 0;$$

thus $\lambda = 4$ or 5, and the general solution for $u(x)$ becomes

$$u(x) = Ax^4 + Bx^5,$$

where *A* and *B* are the arbitrary constants. The solution of the original Riccati equation is therefore

$$y = \frac{u'}{u} = \frac{4Ax^3 + 5Bx^4}{Ax^4 + Bx^5}.$$

This can be recast (and note that this solution is not defined on $x = 0$, so $x \neq 0$) to produce the general solution

$$y(x) = \frac{4 + 5Cx}{x + Cx^2}$$

where *C* ($= B/A$) is a single arbitrary constant.

Comment: If it happens that a (simple) special solution of the Riccati equation is known (so this would be a PI), then it is often convenient to proceed in a different way. We write

$$y(x) = v(x) + \frac{1}{u(x)},$$

where $v(x)$ is the known solution; let us apply this to our general Riccati equation

$$y' = a(x)y^2 + b(x)y + c(x).$$

Substituting for $y(x)$, we obtain

$$\begin{aligned} v' - \frac{u'}{u^2} &= a\left(v + \frac{1}{u}\right)^2 + b\left(v + \frac{1}{u}\right) + c \\ &= a\left(v^2 + \frac{2v}{u} + \frac{1}{u^2}\right) + b\left(v + \frac{1}{u}\right) + c; \end{aligned}$$

but $v(x)$ is a solution of the original equation, so this reduces to

$$-\frac{u'}{u^2} = a\left(\frac{2v}{u} + \frac{1}{u^2}\right) + \frac{b}{u}.$$

Finally, multiplication by u^2 then yields

$$u' + (b + 2av)u = -a$$

which is a standard linear equation (§2.3).

Example 14

Find the general solution of the equation $y' = y^2 - 2xy + x^2 + 1.$

When we write this equation in the form

$$y' = (y - x)^2 + 1$$

it is immediately apparent that there is a solution $y = x$; thus we now seek a solution

$$y = x + 1/u\,.$$

The equation therefore becomes

$$1 - \frac{u'}{u^2} = \left(\frac{1}{u}\right)^2 \text{ i.e. } u' = -1$$

and so $u(x) = A - x$ and then

$$y(x) = x + \frac{1}{A - x}$$

is the required general solution, where A is the arbitrary constant.

3.4 Exact differentials

Let us suppose that we have integrated a first order ODE and produced a solution, $y(x)$, written in the form

$$f(x, y) = A\,,$$

where A is the arbitrary constant. Now we construct a differential equation which has this integral as its solution, by differentiating with respect to x:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' = 0.$$

So, if the original ODE takes this form i.e.

$$a(x,y)+b(x,y)y'=0 \text{ with } a=\frac{\partial f}{\partial x},\ b=\frac{\partial f}{\partial y},$$

for some $f(x,y)$, then we can integrate directly. An equation of this form is called an *exact differential* (or, sometimes, *total differential*). We should observe that, with these definitions of *a* and *b*, we must have

$$\frac{\partial a}{\partial y}=\frac{\partial b}{\partial x};$$

this constitutes a test for an exact derivative.

Example 15

Find the general solution of the equation $\left(x+y^2\right)y'+y=0.$

In this equation, we note that

$$\frac{\partial}{\partial x}\left(x+y^2\right)=1 \text{ and } \frac{\partial}{\partial y}(y)=1$$

and so the equation is an exact differential; indeed, writing

$$\frac{\partial f}{\partial x}=y \text{ and } \frac{\partial f}{\partial y}=x+y^2$$

we see that these imply, respectively, that

$$f(x,y)=xy+F(y) \text{ and } f(x,y)=xy+\tfrac{1}{3}y^3+G(x),$$

where *F* and *G* are arbitrary functions. These two versions of *f* are consistent when we choose $F(y)=\frac{1}{3}y^3-A$ and $G(x)=-A$, where *A* is an arbitrary constant. So there is a general integral of the equation, which defines the solution; it is

$$xy+\tfrac{1}{3}y^3=A.$$

Comment: Even when the ODE is not an exact differential, as given, it may be possible to find a multiplicative factor – an integrating factor – which converts it into an exact form. Thus, given

$$a(x,y)+b(x,y)y'=0,$$

we multiply by $g(x,y)$ to produce

$$g(x,y)a(x,y)+g(x,y)b(x,y)y'=0$$

which will be an exact differential if

$$\frac{\partial}{\partial y}\left[g(x,y)a(x,y)\right]=\frac{\partial}{\partial x}\left[g(x,y)b(x,y)\right],$$

although this may not be easy to solve (for g). Simple cases may, however, turn out to be fairly straightforward.

Example 16

Find the general solution of the equation $2xyy'=x+y^2$.

We observe that $\frac{\partial}{\partial x}(2xy)=2y$ but that $\frac{\partial}{\partial y}\left(-x-y^2\right)=-2y$, so the equation is not exact. Let us therefore construct

$$2xyg(x,y)y'-g(x,y)\left(x+y^2\right)=0$$

and require that

$$\frac{\partial}{\partial x}\left[2xyg\right]=\frac{\partial}{\partial y}\left[-\left(x+y^2\right)g\right]$$

i.e. $$2yg+2xyg_x=-2yg-\left(x+y^2\right)g_y,$$

where the subscripts denote partial derivatives. We can now see that there will be a solution of this equation with $g = g(x)$ only, which then gives

$$2xyg' = -4yg \quad \text{or} \quad xg' = -2g$$

and so we may select the integrating factor $g(x) = 1\big/x^2$. Hence we write the original equation in the form

$$\frac{1}{x^2}\left(x + y^2\right) - \frac{2}{x}yy' = 0 \quad (x \neq 0)$$

which now gives $\dfrac{\partial f}{\partial x} = \dfrac{1}{x} + \dfrac{y^2}{x^2}$ and $\dfrac{\partial f}{\partial y} = -2\dfrac{y}{x}$,

i.e. $f(x,y) = \ln|x| - \dfrac{y^2}{x} + F(y)$ and $f(x,y) = -\dfrac{y^2}{x} + G(x)$ (respectively).

Thus we must choose $F(y) = A$ and $G(x) = \ln|x| + A$ (where A is an arbitrary constant); the general integral of the equation is therefore

$$\ln|x| - \frac{y^2}{x} = -A$$

(which is not defined on $x = 0$). In this example, we are able to write the solution explicitly as

$$y(x) = \pm\sqrt{x\left(A + \ln|x|\right)},$$

which is defined as $x \to 0$.

3.5 Missing variables

We conclude this chapter by considering the equations in which either x or y is absent from the first order ODE $f(x,y,y') = 0$.

First we consider the reduced equation $\phi(y,y') = 0$. The simplest situation that can arise is where this equation can be solved for y', for then

$$y' = \psi(y) \quad \text{and so} \quad x = \int \frac{\mathrm{d}y}{\psi(y)}, \; ,$$

which presents the solution in the form $x = x(y)$; see §5.1.

If, on the other hand, we can solve for y, then we have

$$y = \theta(y') \text{ so } y' = \theta'(y')y'' \text{ i.e. } p = \theta'(p)p'$$

which gives a parametric representation of the general solution:

$$y = \theta(p) \text{ with } x = \int \frac{\theta(p)}{p}\,\mathrm{d}p\,..$$

Example 17

Find the general solution of the equation $y = y' + (y')^2$.

The derivative (with respect to x) of the equation yields

$$y' = (1+2y')y'' \text{ i.e. } \frac{\mathrm{d}p}{\mathrm{d}x} = \frac{p}{1+2p}$$

and so we have $x = \int \frac{1+2p}{p}\,\mathrm{d}p = \ln|p| + 2p + A$

which, with $y = p + p^2$, provides a parametric representation (parameter p) of the general solution (with A the arbitrary constant).

The second case occurs when y is absent: $\phi(x, y') = 0$, and then the most straightforward situation arises when we can write

$$y' = \psi(x) \text{ for then } y = \int \psi(x)\,\mathrm{d}x.$$

If we have $x = \theta(y')$, then the derivative method (used above) can be employed: differentiate with respect to y to give

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \theta'(y')\frac{\mathrm{d}y'}{\mathrm{d}y} \text{ or } \frac{1}{p} = \theta'(p)\frac{\mathrm{d}p}{\mathrm{d}y}$$

and we again develop a parametric representation of the general solution:

$$y = \int p\theta'(p)\,\mathrm{d}p \text{ with } x = \theta(p).$$

Example 18

Find the general solution of the equation $y' + (y')^3 = x$.

The derivative with respect to y gives

$$\left[1+3(y')^2\right]\frac{\mathrm{d}y'}{\mathrm{d}y} = \frac{\mathrm{d}x}{\mathrm{d}y} \text{ or } p(1+3p^2)\frac{\mathrm{d}p}{\mathrm{d}y} = 1$$

and so $y = \frac{1}{2}p^2 + \frac{3}{4}p^4 + A$ which, with $x = p + p^3$, constitutes a parametric representation of the general solution (with A the arbitrary constant).

This concludes the discussion of first-order, ordinary differential equations.

Exercises 3

Find the general solutions of these equations.

(a) $x\left(2x^3 + y\right)y' = 6y^2$ [try $y = w(x^3)$]; (b) $xy' - 3y = x^5 y^{1/3}$;

(c) $y = xy' + \sqrt{1 + (y')^2}$; (d) $y = 2xy' + \ln y'$; (e) $y' + y^2 + 2y + 2 = 0$;

(f) $x^2 y' = x^2 y^2 + xy + 1$ (and one solution is $y = -1/x$);

(g) $\left(x^3 + 2y\right)y' + 3x(xy - 2) = 0$; (h) $x(2 + 3xy)y' + y(3 + 4xy) = 0$;

(i) $y = (y' - 1)\mathrm{e}^{y'}$.

4 Second order ODEs

In this chapter we present methods for solving two standard types of linear, second order ODEs. (However, we do not describe the construction of power-series solutions, using the method of Frobenius.) In addition, we will describe some techniques that are applicable if further information is available (e.g. one solution is known), or the equation takes a special form (e.g. y is absent). Finally, we mention some methods for finding particular integrals of linear equations.

4.1 Constant coefficient equations

We consider the linear equation

$$ay'' + by' + cy = g(x)$$

where a, b and c are given constants. We have already demonstrated (§1.2) that a linear equation possesses a solution which comprises a complementary function (y_{CF}) and a particular integral (y_{PI}); here we concentrate on y_{CF} (which is the general solution of the equation with $g \equiv 0$). In order to develop a general method of solution, we first identify the roots of the quadratic equation

$$a\lambda^2 + b\lambda + c = 0 \;:\; \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \;;$$

let us label these λ_1 and λ_2, and we assume that these are real and distinct. (The other two cases will follow shortly.)

To proceed, let us write (for y_{CF})

$$y'' + \left(\frac{b}{a}\right)y' + \left(\frac{c}{a}\right)y = 0$$

and consider $\left(\frac{\mathrm{d}}{\mathrm{d}x} - \lambda_1\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - \lambda_2 y\right) = 0$; ;

expanding this equation, we obtain

$$y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0$$

where $-(\lambda_1 + \lambda_2) = b/a$ and $\lambda_1\lambda_2 = c/a$: this is our equation for y_{CF} . Thus our second-order equation can be written as the pair of equations

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \lambda_1 u = 0 \text{ with } u = \frac{\mathrm{d}y}{\mathrm{d}x} - \lambda_2 y.$$

The first of these has the general solution $u(x) = A\mathrm{e}^{\lambda_1 x}$ (where A is an arbitrary constant), and then the second equation becomes

$$y' - \lambda_2 y = A\mathrm{e}^{\lambda_1 x}.$$

This equation has an integrating factor (§2.3) $\mathrm{e}^{-\lambda_2 x}$, so we form

$$\mathrm{e}^{-\lambda_2 x}\left(y' - \lambda_2 y\right) = A\mathrm{e}^{(\lambda_1 - \lambda_2)x} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\lambda_2 x} y\right) = Ae^{(\lambda_1 - \lambda_2)x}; \;;$$

thus we obtain $\mathrm{e}^{-\lambda_2 x} y = B\mathrm{e}^{(\lambda_1 - \lambda_2)x} + C$, ,

where *B* and *C* are arbitrary constant. Hence the general solution is

$$y_{CF}(x) = B\mathrm{e}^{\lambda_1 x} + C\mathrm{e}^{\lambda_2 x},$$

which is a general, linear combination of the exponential functions associated with the two roots of the quadratic. We note that *two* arbitrary constants – as expected – have been generated here.

The other two cases can now be examined. First, for repeated roots ($\lambda_1 = \lambda_2$), the only change will be that now we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\lambda_1 x} y\right) = A \quad \text{and so} \quad y(x) = Ax\mathrm{e}^{\lambda_1 x} + B\mathrm{e}^{\lambda_1 x},$$

producing a solution which is no longer purely exponential. The second case occurs when the roots are complex-valued; let us write $\lambda = \alpha \pm \mathrm{i}\beta$, then our solution gives

$$y(x) = \mathrm{e}^{\alpha x}\left(A\mathrm{e}^{\mathrm{i}\beta x} + B\mathrm{e}^{-\mathrm{i}\beta x}\right)$$

for arbitrary (complex) constants *A* and *B*. This is more conveniently written in terms of trigonometric functions:

$$y = e^{\alpha x}\left[A(\cos\beta x + \mathrm{i}\sin\beta x) + B(\cos\beta x - \mathrm{i}\sin\beta x)\right]. .$$

Now we identify new arbitrary constants:

$$C = A + B, \quad D = \mathrm{i}(A - B)$$

to give $y(x) = e^{\alpha x}\left(C\cos\beta x + D\sin\beta x\right)$.

(It is easily confirmed, by direct substitution, that this is indeed a solution of the original equation, for arbitrary *real* constants *C* and *D*. This answers any questions about the validity of working with complex constants and, apparently, a complex-valued solution: in the final analysis, a real, general solution has been constructed.)

The procedure that is typically adopted, in order to find y_{CF} for the constant coefficient equation, is to seek a solution $y = e^{\lambda x}$ (for general *x*). This gives the quadratic equation for λ, and from its solution we are then able to write down the appropriate general solution by following the 'recipe' described above.

Example 19

Find the general solution of the equation $4y'' - 8y' + 3y = 0$.

We set $y = e^{\lambda x}$ to obtain $e^{\lambda x}\left(4\lambda^2 - 8\lambda + 3\right) = 0$; and with $e^{\lambda x} \neq 0$ we require the quadratic expression to be zero. Thus

$$\lambda^2 - 2\lambda + \tfrac{3}{4} = (\lambda - 1)^2 - \tfrac{1}{4} = 0 \quad \text{so} \quad \lambda = 1 \pm \tfrac{1}{2};$$

the general solution is therefore

$$y(x) = Ae^{x/2} + Be^{3x/2},$$

where *A* and *B* are arbitrary constants.

Example 20

Find the general solution of the equation $y'' - 8y' + 16y = 0$.

Set $y = e^{\lambda x}$ then we obtain $\lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0$, so we have a repeated root: $\lambda = 4,$ so the general solution is

$$y(x) = Axe^{4x} + Be^{4x}.$$

Example 21

Find the general solution of the equation $y'' + 4y' + 13y = 0,$ and then that solution which satisfies $y(0) = 1,\ y'(0) = -8.$

Set $y = e^{\lambda x}$ then we obtain $\lambda^2 + 4\lambda + 13 = (\lambda + 2)^2 + 9 = 0$, and so $\lambda = -2 \pm i3$; the general solution is therefore

$$y(x) = e^{-2x}(A\sin 3x + B\cos 3x).$$

Now the given conditions require

$$y(0) = 1 \text{ so } B = 1;\ y'(0) = -8 \text{ so } -8 = -2B + 3A$$

which give $B = 1,\ A = -2$; thus the solution is

$$y(x) = e^{-2x}(\cos 3x - 2\sin 3x).$$

Comment: This same technique is applicable to constant coefficient, linear ODEs of *any* order e.g.

$$y^{iv} + y''' + y'' + y' + y = 0,$$

simply by seeking a solution $y = e^{\lambda x}$ and solving the resulting polynomial equation for $\lambda.$ The general solution is then constructed by following the methods above, and in particular by forming a general, linear combination of all the possible solutions.

4.2 The Euler equation

This equation takes the form

$$ax^2y'' + bxy' + cy = g(x)$$

where *a*, *b* and *c* are given constants. The complementary function is found by following the same philosophy as for the constant coefficient equation, but now based on the function x^λ (rather than $e^{\lambda x}$). Thus we seek a solution $y = x^\lambda,$ to give

$$ax^2\lambda(\lambda - 1)x^{\lambda-2} + bx\lambda x^{\lambda-1} + cx^\lambda = 0,$$

and for this to be valid for any *x*, we require

$$a\lambda(\lambda-1)+b\lambda+c=a\lambda^2+(b-a)\lambda+c=0.$$

The roots for λ then enable the general form of y_{CF} to be found (by adding solutions, just as before, because the ODE is linear); there are three cases.

If the roots are real and distinct (λ_1,λ_2), then we have simply

$$y(x)=Ax^{\lambda_1}+Bx^{\lambda_2}.$$

If the roots are complex: $\lambda=\alpha\pm\mathrm{i}\beta$, then we write

$$y(x)=Ax^{\lambda_1}+Bx^{\lambda_2}.$$

and it is then convenient to use $x^{\pm\mathrm{i}\beta}=e^{\pm\mathrm{i}\beta\ln|x|}=\cos(\beta\ln|x|)\pm\mathrm{i}\sin(\beta\ln|x|)$, , and so we obtain

$$y(x)=x^{\alpha}\left[C\sin(\beta\ln|x|)+D\cos(\beta\ln|x|)\right],$$

for arbitrary real constants *C* and *D*.

Comment: Euler equations and constant coefficient equations are directly and simply related. To see this, we introduce $x=\mathrm{e}^t$ into the Euler equation, and treat $y=y(t)$, which gives

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}=\mathrm{e}^{-t}y'(t)\ \text{ and }\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}\right)\frac{\mathrm{d}t}{\mathrm{d}x}=e^{-2t}(-y'+y''). \quad .$$

Thus our original homogeneous equation becomes

$$a\mathrm{e}^{2t}\mathrm{e}^{-2t}(-y'+y'')+b\mathrm{e}^{t}\mathrm{e}^{-t}y'+cy=0$$

where the prime now denotes the derivative with respect to *t*. The resulting equation is a constant coefficient equation:

$$ay''+(b-a)y'+cy=0.$$

Thus all that we presented in §4.1 can be used e.g. if $y=\mathrm{e}^{\lambda t}$, $\lambda=\lambda_1,\lambda_2$, then

$$y=Ae^{\lambda_1 t}+Be^{\lambda_2 t}=Ax^{\lambda_1}+Bx^{\lambda_2},$$

as obtained above. We may also use this method to obtain the general solution of the Euler equation when the roots are repeated (the third case):

$$y = Ate^{\lambda t} + Be^{\lambda t} = Ax^{\lambda}\ln|x| + Bx^{\lambda}.$$

Example 22

Find the general solution of the equation $x^2 y'' + 2xy' - 6y = 0.$

Set $y = x^{\lambda}$, then we obtain $\lambda(\lambda-1) + 2\lambda - 6 = \lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3) = 0$, so $\lambda = 2, -3$; the general solution is therefore

$$y(x) = Ax^2 + \frac{B}{x^3},$$

where A and B are the arbitrary constants.

4.3 Reduction of order

We now suppose that we have one solution of the homogeneous version of the general, linear second order ODE:

$$a(x)y'' + b(x)y' + c(x)y = 0;$$

we write this solution as $u(x)$ (so that $au'' + bu' + cu = 0$). This assumption is not as extreme as it first seems: we can almost always find at least one solution that contributes to y_{CF} (even if this solution has to be represented as a power series). Now we seek a solution of the original equation in the form $y = u(x)v(x)$, and solve for $v(x)$; we obtain, therefore,

$$a(u''v + 2u'v' + uv'') + b(u'v + uv') + cuv = g$$

which reduces to $auv'' + (2au' + bu)v' = g$.

(This is a version of the 'variation of parameters' – see §4.4 – because the arbitrary constant in $Au(x)$ is treated as a function: the parameter is allowed to vary.) The resulting equation is a general, first-order equation in v' (§2.3) and so the method of solution proceeds as if the order of the equation has been reduced. Note that, in addition to this simplification, the method will also provide the complete solution in that it will produce the appropriate PI that is generated by $g(x)$. It is not normal practice to continue the formulation of the general solution for $v(x)$, and then for $y(x)$; rather, the technique is simply to identify a solution, $u(x)$, and follow the procedure indicated above.

Example 23

Find the complete solution of the equation $\left(1+x^2\right)y'' - 2xy' + 2y = 6\left(1+x^2\right)^2$, given that $y = x$ is a solution of the homogeneous equation.

We first check the given information:

$\left(1+x^2\right).0 - 2x.1 + 2.x = 0$ – which is correct.

Thus we set $y = xv(x)$ to give

$$\left(1+x^2\right)\left(2v' + xv''\right) - 2x\left(xv' + v\right) + 2xv = 6\left(1+x^2\right)^2$$

$$\text{i.e. } x\left(1+x^2\right)v'' + 2v' = 6\left(1+x^2\right)^2.$$

This equation, regarded as an equation for v', is written as

$$v'' + \frac{2}{x\left(1+x^2\right)}v' = \frac{6}{x}\left(1+x^2\right)$$

which possesses the integrating factor $\exp\left\{2\int \frac{\mathrm{d}x}{x\left(1+x^2\right)}\right\}$

$$= \exp\left\{2\int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)\mathrm{d}x\right\} = \exp\left\{2\left[\ln|x| - \tfrac{1}{2}\ln\left(1+x^2\right)\right] + \text{constant}\right\}.$$

We select the integrating factor $\frac{x^2}{1+x^2}$:

$$\left(\frac{x^2}{1+x^2}\right)v'' + \frac{2x}{\left(1+x^2\right)^2}v' = 6x \text{ i.e. } \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{x^2}{1+x^2}\right)v'\right] = 6x$$

and so $\left(\frac{x^2}{1+x^2}\right)v' = 3x^2 + A.$

This gives $v'(x) = A\left(\frac{1+x^2}{x^2}\right) + 3\left(1+x^2\right)$

and then one further integration produces

$$v(x) = A\left(x - \frac{1}{x}\right) + 3x + x^3 + B.$$

Then the complete – because it contains both the CF and the PI – solution of the original equation is

$$y(x) = xv(x) = A\left(x^2 - 1\right) + Bx + 3x^2 + x^4.$$

4.4 Variation of parameters

Here, we expand on the idea developed in the previous section by supposing that we have available both solutions (of a second order, linear ODE) that comprise the CF. (This same method is applicable to any linear, *n*th order ODE, when the CF is completely known.) Consider

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

with two solutions, $u_1(x)$ and $u_2(x)$, of the homogeneous equation given; let us now seek a solution

$$y(x) = A(x)u_1(x) + B(x)u_2(x)$$

(so that both parameters are allowed to vary). Thus we obtain

$$a\left[A''u_1 + 2A'u_1' + Au_1'' + B''u_2 + 2B'u_2' + Bu_2''\right] + b\left[A'u_1 + Au_1' + B'u_2 + Bu_2'\right]$$

$$+c\left[Au_1 + Bu_2\right] = g;$$

but u_1 and u_2 are solutions with $g \equiv 0$, so we obtain

$$a\left[A''u_1 + 2A'u_1' + B''u_2 + 2B'u_2'\right] + b\left[A'u_1 + B'u_2\right] = g.$$

Now we choose $A(x)$ and $B(x)$ such that

$A'u_1 + B'u_2 = 0$ (so that $A''u_1 + A'u_1' + B''u_2 + B'u_2' = 0$)

to leave $a(x)\left[A'u_1' + B'u_2'\right] = g(x)$,

which together provide two equations for $A'(x)$ and $B'(x)$. The solution of these equations is

$$A' = -\frac{g}{a}\frac{u_2}{W}, \quad B' = \frac{g}{a}\frac{u_1}{W}$$

where $W(x) = u_1u_2' - u_1'u_2$ ($\neq 0$ for linearly independent solutions) is called the *Wronskian*. The general solution of the original equation is now expressed as

$$y(x) = -u_1(x)\left(\int \frac{g(x)u_2(x)}{a(x)W(x)}\,\mathrm{d}x\right) + u_2(x)\left(\int \frac{g(x)u_1(x)}{a(x)W(x)}\,\mathrm{d}x\right).$$

Example 24

Find the complete solution of the equation $y'' + y = \sin x$.

The general solution of the homogeneous equation is

$$y = A\sin x + B\cos x,$$

and so we select $u_1(x) = \sin x, u_2(x) = \cos x,$ and then

$$W(x) = -\sin^2 x - \cos^2 x = -1.$$

Thus we have the complete solution expressed as

$$\begin{aligned}
y(x) &= \sin x\left(\int \sin x\cos x\,\mathrm{d}x\right) - \cos x\left(\int \sin^2 x\,\mathrm{d}x\right)\\
&= \sin x\left(\int \tfrac{1}{2}\sin 2x\,\mathrm{d}x\right) - \cos x\left(\int \tfrac{1}{2}(1-\cos 2x)\,\mathrm{d}x\right)\\
&= \sin x\left(-\tfrac{1}{4}\cos 2x + A\right) - \cos x\left(\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + B\right)\\
&= A\sin x + B\cos x - \tfrac{1}{2}x\cos x + \tfrac{1}{4}(\sin 2x\cos x - \cos 2x\sin x)\\
&= A\sin x + B\cos x - \tfrac{1}{2}x\cos x + \tfrac{1}{4}\sin x.
\end{aligned}$$

4.5 Finding particular integrals directly

For general, linear equations of second order:

$$a(x)y'' + b(x)y' + c(x)y = g(x),$$

we can always find the PI by using one form (§4.3) or another (§4.4) of the method of variation of parameters. However, if $g(x)$ takes a simple form – especially in the case of constant-coefficient equations – then it is possible to find the PI directly by, essentially, algebraic means. We will briefly present systematic approaches in three simple situations (but obvious extensions of these can also be tackled using these techniques). Of course, it is always possible, when seeking PIs, to make an appropriate guess and confirm its correctness, but we opt for something more methodical here.

(a) *g*(*x*) is a polynomial

In this case, we differentiate the original equation as many times as is necessary to reduce $g(x)$ to a non-zero constant. A form of the PI then follows by solving the equations, starting from the simplest solution – usually a suitable constant – of the final equation.

Example 25

Find a particular integral of the equation $x^2y'' + (1+2x)y' + 3y = x^2$.

We form

$$2xy'' + x^2 y''' + 2y' + (1+2x)y'' + 3y' = 2x$$

and then $2y'' + 4xy''' + x^2 y^{iv} + 4y'' + (1+2x)y''' + 3y'' = 2;$

this last equation has a simple solution with $y''' = 0$ and $y^{iv} = 0$ i.e. $y'' = 2/9$. Then the preceding equation gives

$$(1+4x)\tfrac{2}{9} + 5y' = 2x \text{ or } y' = \tfrac{2}{9}x - \tfrac{2}{45},$$

and finally the original equation becomes

$$x^2\,\tfrac{2}{9} + (1+2x)\left(\tfrac{2}{9}x - \tfrac{2}{45}\right) + 3y = x^2 \text{ so } y_{PI}(x) = \tfrac{1}{9}x^2 - \tfrac{2}{45}x + \tfrac{2}{135}.$$

(b) $g(x)$ is a polynomial $\times$ exponential

We now have an equation that takes the form

$$a(x)y'' + b(x)y' + c(x)y = P(x)\mathrm{e}^{\alpha x}$$

where $P(x)$ is a given polynomial. The most direct way to find the PI is to seek a solution in the form $y(x) = u(x)\mathrm{e}^{\alpha x}$, which will reduce the problem to finding a suitable $u(x)$ following the procedure in case (a).

Example 26

Find a particular integral of the equation $y'' + y' - 2y = x\mathrm{e}^{-2x}$.

We set $y = u(x)\mathrm{e}^{-2x}$, which gives

$$\left(u'' - 4u' + 4u\right)\mathrm{e}^{-2x} + \left(u' - 2u\right)\mathrm{e}^{-2x} - 2u\mathrm{e}^{-2x} = x\mathrm{e}^{-2x}$$

and this simplifies to $u'' - 3u' = x$.

Thus we obtain, after one differentiation, $u''' - 3u'' = 1$

which has the simple solution $u'' = -1/3$ ($u''' = 0$), and so we then have

$-\tfrac{1}{3} - 3u' = x$ and then we choose $u(x) = -\tfrac{1}{6}x^2 - \tfrac{1}{9}x$

(and the arbitrary constant is irrelevant for a PI); hence

$$y_{PI}(x) = -\tfrac{1}{18}\left(3x^2 + 2x\right)\mathrm{e}^{-2x}.$$

(c) *g*(*x*) is a polynomial $\times \sin(\alpha x)$ **(or** $\cos(\alpha x)$ **)**

The equation is now

$$a(x)y'' + b(x)y' + c(x)y = P(x)\sin(\alpha x)$$

(or $= P(x)\cos(\alpha x)$ or $= P(x)\sin(\alpha x) + Q(x)\cos(\alpha x)$, where $P(x)$ and $Q(x)$ are polynomials). This time we seek a solution

$$y_{PI}(x) = u(x)\sin(\alpha x) + v(x)\cos(\alpha x)$$

and then obtain two problems, one for each of $u(x)$ and $v(x)$ (although these equations, in general, will be coupled); the procedure again follows that in case (a). As before, this method is especially relevant when the differential equation has constant coefficients.

Example 27

Find a particular integral of the equation $y'' + y' - 2y = x\sin 2x + \cos 2x$.

We seek a solution

$$y_{PI}(x) = u(x)\sin 2x + v(x)\cos 2x$$

and then we obtain

$$u''\sin 2x + 4u'\cos 2x - 4u\sin 2x + v''\cos 2x - 4v'\sin 2x - 4v\cos 2x$$

$$+u'\sin 2x + 2u\cos 2x + v'\cos 2x - 2v\sin 2x - 2u\sin 2x - 2v\cos 2x = x\sin 2x + \cos 2x.$$

Eliminating the terms, separately, in $\sin 2x$ and $\cos 2x$, yields

$$u'' + u' - 4v' - 6u - 2v = x\,;\; v'' + v' + 4u' - 6v + 2u = 1.$$

We must now proceed with a little care; first, we eliminate either u or v (but not necessarily the corresponding derivatives) to give e.g.

$$u'' + 3v'' + 13u' - v' - 20v = x + 3$$

and the derivative of this produces

$$u''' + 3v''' + 13u'' - v'' - 20v' = 1$$

which has a special solution $v' = -1/20$ (with $v'' = u'' = v''' = u''' = 0$). The derivative of our first equation gives

$$u''' + u'' - 4v'' - 6u' - 2v' = 1,$$

so we have a solution $u' = (-1 + \tfrac{1}{10})\big/6 = -3/20$. The two original equations now require

$$-\tfrac{3}{20} + \tfrac{1}{5} - 6u - 2v = x\,;\; -\tfrac{1}{20} - \tfrac{3}{5} + 2u - 6v = 1$$

which, upon solution, give $u(x) = -\tfrac{3}{20}x + \tfrac{33}{40}$; $v(x) = -\tfrac{1}{20}x - \tfrac{49}{200}$.

Thus a particular integral is

$$y_{PI}(x) = -\frac{x}{20}(3\sin 2x + \cos 2x) + \frac{1}{200}(165\sin 2x - 49\cos 2x).$$

4.6 Missing variables

General second-order equations

$$f(x, y, y', y'') = 0$$

will simplify, and usually succumb to one of our earlier methods, if a suitable variable happens to be missing. The simplest situation occurs if y is absent, for then the equation becomes of *first order* in y'.

Example 28

Find the general solution of the equation $x^2 y'' - (y')^2 + 2xy' = 2x^2$.

First, we set $p(x) = y'$, and so obtain

$$x^2 p' = p^2 - 2xp + 2x^2$$

which is a homogeneous, first-order equation (see §2.2). Thus we now write $p = xv(x)$:

$$xv' + v = v^2 - 2v + 2 \text{ or } xv' = v^2 - 3v + 2 = (v-1)(v-2)$$

$$\text{i.e. } \int \frac{\mathrm{d}v}{(v-1)(v-2)} = \int\left(\frac{1}{v-2} - \frac{1}{v-1}\right)\mathrm{d}v = \int \frac{\mathrm{d}x}{x}.$$

So we obtain $\ln\left|\dfrac{v-2}{v-1}\right| = \ln|x| + A$ or $\dfrac{v-2}{v-1} = Bx$ i.e. $v = \dfrac{Bx-2}{Bx-1}$

and hence $y' = xv = x\left(\dfrac{Bx-2}{Bx-1}\right) = x\left(1 - \dfrac{1}{Bx-1}\right) = x - \dfrac{1}{B} - \dfrac{1}{B}\dfrac{1}{(Bx-1)}$.

One final integration then produces the general solution

$$y(x) = \tfrac{1}{2}x^2 - \tfrac{1}{B}x - \tfrac{1}{B^2}\ln|Bx-1| + C$$

where B and C are the arbitrary constants.

The second case that we will address is when x is absent (but otherwise we have a general equation); here we may treat $y' = p(y)$ i.e. $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}p}{\mathrm{d}y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = p\dfrac{\mathrm{d}p}{\mathrm{d}y}$, so

$$\phi(y, y', y'') = 0 \text{ becomes } \phi(y, p, pp') = 0:$$

a first-order equation for $p(y)$.

Example 29

Find the general solution of the equation $yy'' + (y')^2 = 1$.

Set $p = y'$, then $y'' = pp'(y)$ and so we obtain

$$ypp' = 1 - p^2$$

which is separable (§2.1):

$$\int \frac{p\mathrm{d}p}{1-p^2} = \int \frac{\mathrm{d}y}{y} \text{ so } -\tfrac{1}{2}\ln\left|1-p^2\right| = \ln|y| + A.$$

Thus $\quad y^2\left(1-p^2\right) = B$ and then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \pm\dfrac{1}{y}\sqrt{y^2-B}$

i.e. $\quad \displaystyle\int \frac{y}{\sqrt{y^2-B}}\,\mathrm{d}y = \pm\int \mathrm{d}x$ so $\sqrt{y^2-B} = \pm x + C.$

The general solution is $y(x) = \pm\sqrt{B+(x+D)^2}$,

where B and D are arbitrary constants.

Exercises 4

Find the general solutions of these equations.

(a) $y'' + 3y' - 10y = 0$; (b) $4y'' + 20y' + 25y = 0$;
(c) $y'' + 2y' + 5y = 0$; (d) $x^2y'' - 3xy' + 8y = 0$;
(e) $\left(1+x^2\right)y'' + xy' - y = 2 + 3x^2$ where one solution of the homogeneous equation is $y = x$; (f) $x^2y'' + xy' - 4y = x^2 + x^4$ (use variation of parameters);
(g) $y'' - 3y' + 2y = x^3$; (h) $y'' - 3y' + 2y = x\mathrm{e}^x + \mathrm{e}^{2x}$;
(i) $y'' + y = x\sin x$; (j) $xy'y'' = x^4 + (y')^2$; (k) $yy'' = y' + (y')^2$.

5 More general aspects of ODEs

In this final chapter, we touch on three rather more general ideas that those who seek some deeper ideas may find informative. We will extend, a little, the idea that we met in §3.2 and §3.5, where the roles of x and y are interchanged. Then we will highlight the important fact that some equations admit exceptional solutions – and we have already met a few – but now we put this possibility within a systematic framework of solutions. We conclude by making some observations about the nature of uniqueness proofs; with uniqueness available, we can obtain (as examples) two important identities between the trigonometric functions.

5.1 Interchanging x and y

This idea has been encountered in equations of the type $\phi(y, y') = 0$ (at the start of §3.5) which take the particular form

$$y' = \psi(y) \text{ so that } x = \int \frac{\mathrm{d}y}{\psi(y)}.$$

There is, however, another important class of equations that also falls into this category (and we limit ourselves to first order ODEs); these equations take the form

$$[xf(y) + g(y)]y' = h(y).$$

This equation is of order 1 and degree 1 but, in general, it is nonlinear – maybe highly so, and cannot be integrated (in the general case) by any of our methods, as it stands. However, if we elect to treat $x = x(y)$ rather than $y = y(x)$, we may write

$$h(y)\frac{\mathrm{d}x}{\mathrm{d}y} - xf(y) = g(y)$$

which is a standard, linear equation (§2.3) in *x*.

Example 30

Find the general solution of the equation $\left(x + y^2 \mathrm{e}^y\right)y' = y$.

We write the equation as

$$x'y - x = y^2\mathrm{e}^y \text{ or } x' - \frac{1}{y}x = y\mathrm{e}^y \quad (y \neq 0)$$

which has the integrating factor $\exp\left(-\int \frac{\mathrm{d}y}{y}\right) = \exp\left(-\ln|y| + A\right)$, , so we elect to use y^{-1}:

$$\frac{1}{y}x' - \frac{x}{y^2} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{x}{y}\right) = e^y.$$

Thus $x/y = \mathrm{e}^y + A$ and so the general solution can be expressed by the relation

$$Ay + y\mathrm{e}^y = x,$$

where *A* is an arbitrary constant (and $y = 0$ simply implies $x = 0$: a point).

5.2 Singular solutions

In this section we comment on a special solution which may, exceptionally, be the relevant solution to our first order ODE. Some of the early examples (in Chapters 2 and 3) described differential equations that had special solutions that had to be checked against the general solution, once it had been obtained. In all but two cases, these solutions, it transpired, could be recovered by taking special values (or limits) of the arbitrary constant; thus, eventually, they were not exceptional at all. However, in Example 5 (§2.2) and Example 11 (§3.2), we found exceptional solutions that could *never* be recovered from the general solution. Any solution of the differential equation which cannot be obtained from the general solution is called a *singular solution*. It turns out that a singular solution *is* intimately associated with the general solution: it is the *envelope* of the general solution, mapped out as the arbitrary constant varies. We will explain how this comes about.

Let $f(x, y, A) = 0$ be the general solution of a first order ODE, where *A* is the arbitrary constant. Now a singular solution has the same slope as the general solution, at the same point, because this is what the ODE implies. In addition, it also coincides – at some point – with consecutive curves of the general solution, generated as *A* varies from A_1 to A_2 (say), but in the limit as $A_1 \to A_2$. This is the definition of an *envelope* of a family of curves; it requires that $f(x, y, A) = 0$

has a double root in A i.e. $\frac{\partial f}{\partial A} = 0$. The envelope (if one exists) is therefore the solution(s) obtained by eliminating A between

$$f(x, y, A) = 0 \text{ and } \frac{\partial}{\partial A} f(x, y, A) = 0.$$

This property is satisfied by our Example 11, where the general solution was given as

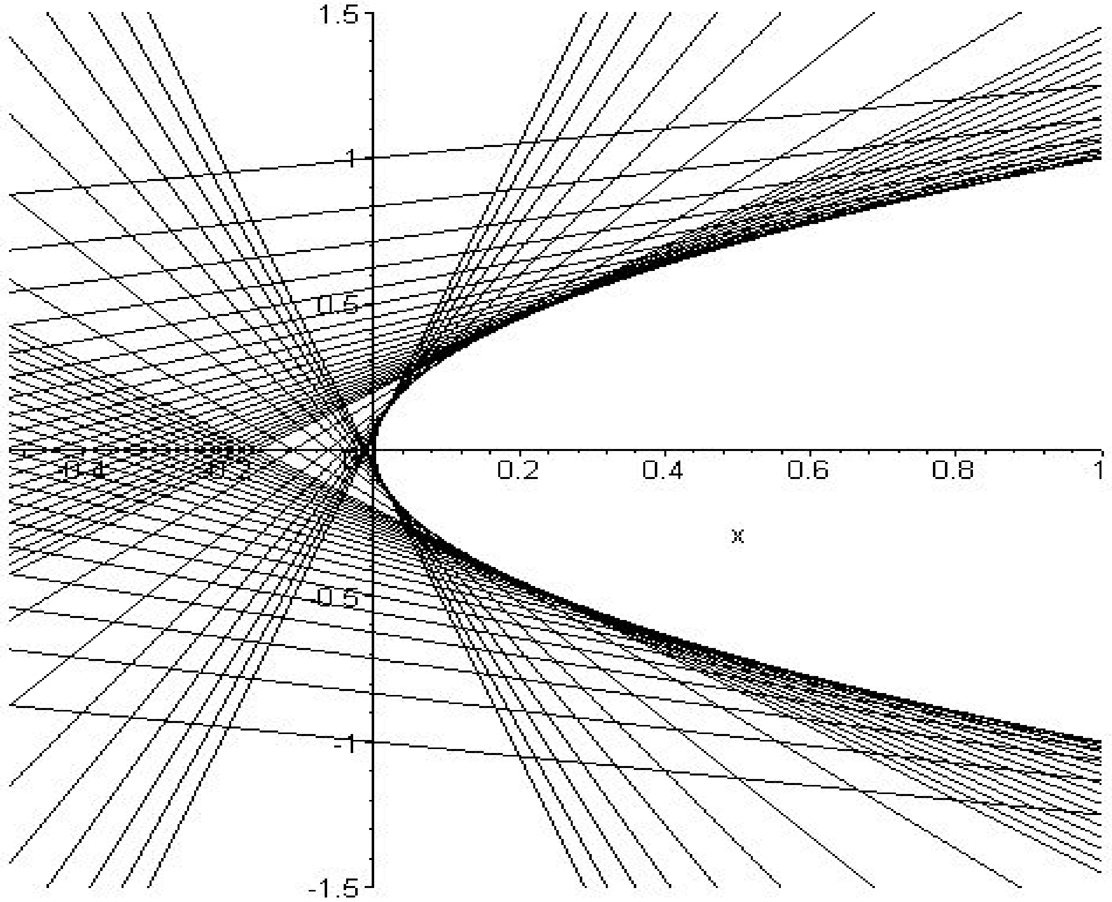
$$f(x, y, A) \equiv y - \left(Ax + \frac{1}{4A} \right) = 0,$$

$$\text{and so } \frac{\partial f}{\partial A} = -x + \frac{1}{4A^2} = 0;$$

thus the envelope is described by

$$(Ay)^2 = \left(A^2 x + \tfrac{1}{4} \right)^2 \text{ with } A^2 = 1/4x \text{ i.e. } y^2 = x.$$

This is the exceptional solution that we found in Example 11; a selection of straight lines – the general solution – and the envelope are depicted in the figure below.



Comment: Any singular solution of the differential equation can be determined directly from the equation simply by selecting the condition that y' has a repeated root (since, as we have seen, at least two solutions exist with the same slope, at some point). In Example 11: $x(y')^2 - yy' + \frac{1}{4} = 0$, so y' has a repeated root if $(-y)^2 = 4.x.\frac{1}{4}$ i.e. $y^2 = x$, as previously obtained.

Example 31

Find the envelope of the general solution $y(x) = \dfrac{1}{2A}\left(1 + A^2x^2\right)$ (as derived in Example 5).

Given $f(x, y, A) \equiv y - \left(\dfrac{1}{2A} + \dfrac{A}{2}x^2\right) = 0$, then

$$\frac{\partial f}{\partial A} = \frac{1}{2A^2} - \frac{1}{2}x^2 = 0 \text{ so } A^2x^2 = 1,$$

and eliminating A produces

$$y = \frac{1}{2A}\left(1 + A^2x^2\right) = \frac{1}{A} \text{ or } y^2 = A^{-2} = x^2;$$

the envelopes are $y = \pm x$, the exceptional solutions already reported in Example 5.

Comment: We can readily see that solutions of linear equations can never generate envelope solutions, and so singular solutions of these equations do not exist. Consider the solution of some linear equation; it will take the general form

$$y(x) = A\alpha(x) + \beta(x)$$

for some functions α and β, where A is the arbitrary constant. Thus we have, for any envelope curve,

$$f(x, y, A) \equiv y - A\alpha(x) - \beta(x) = 0 \text{ and so } \frac{\partial f}{\partial A} = -\alpha(x) = 0$$

which gives $y(x) = \beta(x)$, which is the general solution with $A = 0$: there is no exceptional 'envelope' solution. Of course, we could have simply noted that a solution which is linear in A (or, equivalently, a differential equation linear in y') cannot have a repeated root for y'.

5.3 Uniqueness

We have seen, in §5.2, that some ODEs have more than one solution through certain points in the (x,y)-plane. This observation prompts us to consider the general issue of uniqueness; perhaps solutions of ODEs suffer from non-uniqueness, in a more general sense. (The question of existence is not one that we will pursue here – we have, after all, derived solutions of many equations – but this may be a critical issue if solutions cannot be constructed by any method. In this situation, a numerical solution may be considered, but this would be a wasteful exercise if we do not have some conviction that a suitable solution exists.) We will, in all that follows, consider only first order ODEs.

Most uniqueness theorems assume, first, that the ODE can be expressed as $y' = f(x, y)$ and, secondly, that f satisfies a *Lipschitz condition* in y:

$$\left| f(x, y_1) - f(x, y_2) \right| \le k \left| y_1 - y_2 \right|,$$

where k is a (positive) constant, for all functions which may be solutions. We now consider the ODE with the boundary condition $y(a) = b$; let us suppose that there are two solutions, $y_1(x)$ and $y_2(x)$, that satisfy both the ODE and the given boundary condition. We form

$$y_1' - y_2' = f(x, y_1) - f(x, y_2)$$

and then apply the Lipschitz condition (which, of course, we assume applies here), to give

$$\left| y_1' - y_2' \right| \le \left| f(x, y_1) - f(x, y_2) \right| \le k \left| y_1 - y_2 \right| \text{ i.e. } \left| w' \right| \le k \left| w \right|,$$

where $w(x) = y_1(x) - y_2(x)$. Thus (provided $|w| \ne 0$ at this stage) we obtain

$$\left| (\ln|w|)' \right| \le k \text{ or } A\mathrm{e}^{-k(x-a)} \le w \le B\mathrm{e}^{k(x-a)},$$

where A and B are arbitrary constants; but $w = 0$ on $x = a$, so at equality within the inequalities we have $A = B = 0$ i.e. $w \equiv 0$. The solution of the ODE is unique.

In particular cases, we can often use this argument directly, without recourse to the introduction of a general Lipschitz condition.

Example 32

Construct a uniqueness theorem for the general Riccati equation $y' = a(x)y^2 + b(x)y + c(x)$; see §3.3.

Let two solutions that satisfy $y(x_0) = y_0$ be $y_1(x)$ and $y_2(x)$, then we have

$$(y_1 - y_2)' = a(x)\left(y_1^2 - y_2^2\right) + b(x)(y_1 - y_2)$$

or $$w' = a(x)(y_1 + y_2)w + b(x)w = \left[a(x)(y_1 + y_2) + b(x)\right]w,$$

where $w(x) = y_1(x) - y_2(x)$. Thus we may integrate:

$$w(x) = A\exp\left\{\int_{x_0}^{x}\left[a(x')v(x') + b(x')\right]\mathrm{d}x'\right\},$$

where $v(x) = y_1(x) + y_2(x)$ and A is an arbitrary constant; the integral, and therefore $w(x)$, exist if $y_1(x)$ and $y_2(x)$ do (and we know that solutions do exist, at least for suitable $a(x),\ b(x)$ and $c(x)$). But $w = 0$ on $x = x_0$, so $A = 0$: $w \equiv 0$ i.e. the solution that exists is unique.

We conclude by invoking uniqueness when we solve our final pair of ODEs; in each, we solve in two different ways and set the two representations of the solution equal.

Example 33

Find the general solution of the Riccati equation $y' = 1 + y^2$, expressed in two different ways (and then equate them).

The standard Riccati approach is to set, in this example, $y = -\phi'/\phi$, to give

$$-\left(\frac{\phi''}{\phi} - \frac{(\phi')^2}{\phi^2}\right) = 1 + \left(-\frac{\phi'}{\phi}\right)^2 \text{ i.e. } \phi'' + \phi = 0,$$

and so $\phi(x) = A\sin x + B\cos x$.

The general solution is thus $y(x) = -\dfrac{\phi'}{\phi} = \dfrac{C + \tan x}{1 - C\tan x}$,

where $C\ (= -A/B)$ is the arbitrary constant.

However, this equation can be integrated directly – it is separable – as

$$\int \frac{\mathrm{d}y}{1+y^2} = \int \mathrm{d}x \ \text{ so } \ \arctan(y) = x + D \ \text{ i.e. } \ y(x) = \tan(x+D).$$

The uniqueness of the solution (which we have demonstrated for this equation in Example 32) implies that

$$\tan(x+D) = \frac{C + \tan x}{1 - C\tan x} \quad (\forall x)$$

and we note that, on $x = 0$, we require $\tan D = C$, so we obtain

$$\tan(x+D) = \frac{\tan x + \tan D}{1 - \tan x . \tan D}$$

– a very familiar identity.

Example 34

Find the general solution of the equation $y' = -\sqrt{\dfrac{1-y^2}{1-x^2}}$ (usually called *Euler's equation*) in two different forms, and write down the condition that implies that the two solutions are the same solution.

The equation is separable, so we have

$$\int \frac{\mathrm{d}y}{\sqrt{1-y^2}} = -\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} \text{ so } \arcsin y + \arcsin x = A,$$

where A is an arbitrary constant.

However, a little investigation shows that this equation also possesses an integrating factor:

$$\left\{\sqrt{\left(1-y^2\right)\left(1-x^2\right)} - xy\right\}\left(\frac{y'}{\sqrt{1-y^2}} + \frac{1}{\sqrt{1-x^2}}\right) = 0$$

which gives $x\sqrt{1-y^2} + y\sqrt{1-x^2} = B,$

where B is a second arbitrary constant. It is rather tiresome (and altogether unnecessary) to solve explicitly for $y(x)$, so we will invoke uniqueness by observing that the two $y(x)$s being identical implies that the two constants must be related i.e. $A = F(B)$:

$$\arcsin y + \arcsin x = F\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

for some function *F*. But for $y = 0$ we obtain

$$\arcsin x = F(x),$$

which leads to

$$\arcsin y + \arcsin x = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right).$$

Let us introduce $\alpha = \arcsin x$ and $\beta = \arcsin y$, then our identity becomes

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

another very familiar identity.

Exercises 5

Find the general solutions, and any singular solutions, of these equations.

(a) $\left(y^2+2y-3x\right)y'+y=0$; (b) $x(y')^2-2yy'+4x=0$;

(c) $(y')^2-xy'-y=-\frac{1}{2}x^2$; (d) $(y')^2-yy'=-\mathrm{e}^x$;

(e) $(y')^2-yy'=\mathrm{e}^x$; (f) $(y')^3-4xyy'+8y^2=0$.

Answers

Exercises 1

(a) order 1, degree 1, nonlinear; (b) order1, degree 1, linear; (c) order 1, degree 2;

(d) order 2, degree 1, linear; (e) order 2, degree 1, linear;

(f) order 2, degree 1, nonlinear; (g) order 3, degree 1, nonlinear;

(h) order 4, degree 1, linear.

Exercises 2

The arbitrary constant is *A* in each case.

(a) $y(x) = \mathrm{e}^{A/x}$; (b) $y(x) = \ln\left|A\left(1+x^2\right)-1\right|$; (c) $y\mathrm{e}^{-y/x} = A$;

(d) $y = A\left(1+\exp\left[(x/y)^2\right]\right)$ and then $y(0) = 4$ requires $A = 2$;

(e) $y(x) = x^2(A+\sin x)$; (f) $y(x) = A\mathrm{e}^{-x}\left(1+x^2\right) - \frac{1}{2}\mathrm{e}^{-x}$.

Exercises 3

The arbitrary constant is A in each case; the parameter is p.

(a) $y(x) = 2x^3 + Ax^6 \pm x^3\sqrt{\left(2+Ax^3\right)^2 - 4}$; (b) $y(x) = \left(Ax^2 + \frac{2}{9}x^5\right)^{3/2}$;

(c) $y(x) = Ax + \sqrt{1+A^2}$; $y(x) = \sqrt{1-x^2}$;

(d) $x = Ap^{-2} - p^{-1}$, $y = 2Ap^{-1} + \ln|p| - 2$; (e) $y(x) = -1 + \dfrac{\cos x - A\sin x}{\sin x + A\cos x}$;

(f) $y(x) = -\dfrac{1}{x} + \dfrac{1}{x(A - \ln|x|)}$; (g) $y(x) = -\frac{1}{2}x^3 \pm \sqrt{A + 3x^2 + \frac{1}{4}x^6}$;

(h) $x^3y^2 + x^4y^3 = A$ (int. fact. x^2y); (i) $y(x) = (x-A)\left(\ln|x-A| - 1\right)$, $y(x) = -1$.

Exercises 4

The arbitrary constants are A and B in each case.

(a) $y(x) = Ae^{2x} + Be^{-5x}$; (b) $y(x) = Axe^{-5x/2} + Be^{-5x/2}$;

(c) $y(x) = e^{-x}(A\sin 2x + B\cos 2x)$; (d) $y(x) = x^2\left(A\sin(2\ln|x|) + B\cos(2\ln|x|)\right)$; ;

(e) $y(x) = Ax + B\sqrt{1+x^2} + x^2$; (f) $y(x) = Ax^2 + Bx^{-2} + \frac{1}{4}x^2\ln|x| + \frac{1}{12}x^4$;

(g) $y(x) = Ae^{x} + Be^{2x} + \frac{1}{2}x^3 + \frac{9}{4}x^2 + \frac{21}{4}x + \frac{45}{8}$;

(h) $y(x) = Ae^{x} + Be^{2x} + xe^{2x} - \left(\frac{1}{2}x^2 + x\right)e^{x}$; ;

(i) $y(x) = A\sin x + B\cos x + \frac{1}{4}x(\sin x - x\cos x)$;

(j) $y(x) = A \pm \frac{1}{3}\sqrt{\left(x^2+B\right)^3}$; (k) $y(x) = \dfrac{1}{A}\left(1 + Be^{Ax}\right)$.

Exercises 5

The arbitrary constant is A in each case; any singular solution is quoted separately.

(a) $x(y) = Ay^3 + y^2 + y$; (b) $y(x) = A + \dfrac{x^2}{A}$, $y(x) = \pm 2x$;

(c) $y(x) = Ax + A^2 + \frac{1}{2}x^2$, $y(x) = \frac{1}{4}x^2$; (d) $y(x) = Ae^{x} + A^{-1}$, $y(x) = \pm 2e^{x/2}$;

(e) $y(x) = Ae^{x} - A^{-1}$ (only); (f) $y(x) = A(x-A)^2$, $y(x) = \frac{4}{27}x^3$.

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